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Mathematics at Work in the Paper Industry*

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IF WE WERE GATHERED here to discuss Latin instead of mathematics, that is, if you were teachers of Latin instead of mathematics teachers, the purpose of my talk might be to reassure you that, "yes," Latin is a useful subject; that, "yes," it is employed in industry; that, "yes," you people are serving a useful purpose in teaching the subject. But since, instead, you are teachers of mathematics, and since we will all agree that mathematics is here to stay, the purpose of my talk can be only to show what kinds of mathematics are useful in one of America's major industries. Unless mathematics is to be taught strictly as a mental exercise and discipline, it is probably well to survey all the major industries with a view toward emphasizing in school the mathematics which is most important in industry and which will be used the most in actual life. Since I doubt that there are any rare forms of mathematics or highly unusual mathematical techniques peculiar to the paper industry, I can, to a degree, represent industry in general; but strictly as a papermaker, I represent a rather considerable segment of American industry. The manufacture of paper and paperboard is America's sixth largest industry. Last year we produced 24,000,000 tons of paper and paperboard in this country, a total of over 300 pounds for each man, woman, and child.

Although paper has been made for thousands of years, it is only in recent years that we have begun to speak of papermaking as a science rather than an art. However, under modern competitive conditions, with the demand for more and more production and the demand for greater and greater efficiency, and with the increase of scientific knowledge which is characteristic of modern life, the trend toward science and away from art has been moving very fast, and the pace is accelerating. What does this mean? Whereas in the old days a worker was guided by subjective judgment based on long years of experience—he put in more or less alum, or used more or less pressure in stock treatment, or turned steam on or off according to rule of thumb, if indeed he could quote any rule at all—nowadays, we are approaching the point of having a reason and a formula for everything with everything on a quantitative basis. More and more, the man in the mill finds that he must be able to handle figures and formulas if he is going to hold his job or rise to the job above. I might add here that this is over and above the ability to figure out a pay check—no mean feat in itself, in these days of time and one-half for overtime, double time for Sundays, triple time for legal holidays, less deductions for

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taxes, for bonds, for insurance, etc., etc.

The leaders of this trend toward science have been, of course, the engineers, the chemists, the development men, that is, the technically trained men—usually products of our colleges and universities, about four thousand of whom are members of the Technical Association of the Pulp and Paper Industry. For them, mathematics is an absolutely essential tool. In the very extensive technical literature of the pulp and paper industry, there is a liberal sprinkling of integral signs and differential equations.

Paper is made on some of the largest and most complicated machines in all industry, often costing over a million dollars apiece. The design, building, and installation of these machines is an industry in itself, employing many mechanical and structural engineers, with all the necessary mathematics at their command. Many paper companies design a large part of their own equipment and buildings and have their own rather extensive staffs of mechanical and structural engineers as permanent employees. A number of Ph.D.'s in physics as permanent employees are aiding in this design work.

Since paper is made from a fluid suspension, subject to turbulence, precise control of that suspension at high speed becomes a major problem. The physicists not only employ the equations of hydrodynamics but do research in that field. I personally know of one paper company which has a large wind tunnel among its research facilities, because air is a fluid and follows many of the same laws that influence the flow of liquids.

The physicists also have their say in respect to instrumentation. The faster a process becomes, the more difficult it becomes to observe and control through human attention; the more automatic the process becomes, the greater becomes the need for more instruments. We are not yet at the push-button stage in the paper industry where a log is put in at one end of a machine and finished paper comes out at the other end, with the attention of only one operator sitting at a console of switches. One look at a modern instrument panel, which may be twenty feet long, may make one think, though, that the paper industry is well on its way toward that push-button stage.

Now suppose we have a beautifully engineered pulp mill and a brand-new $2,000,000 paper machine with an impressive switchboard, ten feet high, with green, red, and amber lights flashing on and off. Does this mean that we can take for granted that the product will be good and, without looking at it, ship it off for printing *National Geographic* magazines or for making paper cups or woven fabric for automobile seat covers? "No!" Raw materials are variable, and specifications that the product must meet are exacting. Day-to-day weather differences affect the production of paper. Normal wear of the machinery parts affects the quality. Human errors may creep in and, in some cases, effects occur which are not fully understood. Call them "gremlins" if you will.

As I said before, the push-button era is not yet here and, as a result, there must be continual sampling and testing of the product and comparisons made with specified standards. Every mill has its paper testers. The manufacturing machines run twenty-four hours a day, and ordinarily the testers are there all the time. They work with tests, data, and reports thereof. More will be said later about statistical quality control.

Now who are they who set these specifications and why are these standards so exacting? To many of us, paper is just something to be glanced at or, perhaps, scribbled on and quickly thrown away, but to the man who buys it, particularly those who buy it in large quantities, it represents good, hard money. The buyer is acutely aware of the differences in quantity and quality that he gets for his money. This is particularly true if he is a printer. Paper is the printer's major raw

material, directly affecting the quality of his work, his reputation, and the profits he makes. The weight of paper must be closely controlled. If it runs heavy, the buyer does not get as many square feet of surface per pound of paper as he should. On the other hand, if the paper runs light, the paper mill is losing money because it takes longer for a paper machine to run out the required number of pounds of paper. Light-weight paper is often too thin, too weak, or not sufficiently opaque. Heavy paper is often too thick or too stiff. Variation of weight across the length or width of a sheet is particularly bad. A printer sets his press for paper of a certain thickness. If one sheet is thick and the next thin, he has intolerable variations. If a sheet is thinner in the middle or along one edge, it is impossible to do a uniformly good printing job all over the sheet. Converting machines cannot operate rapidly and efficiently unless weight and thickness and other properties, such as stiffness, are held to close tolerances.

Paper is always made to order. Even if it is a stock item intended to be sold by stationers, a pound or so at a time, it still is made in accordance with somebody's definite ideas and with definite uses in mind. Mimeograph paper is not the same as typewriter bond. Paper for ordinary letterpress printing might look very much like paper for offset lithography but there is a definite difference. All the way from facial tissue to the heaviest boxboard, every sheet of paper is designed for a purpose. The more scientific the papermaker becomes, the more physical principles he learns and the more he realizes where these principles apply and how to apply them. This process leads to a knowledge of how to state exact mathematical specifications for papermaking.

In respect to the property of stiffness, the same mathematical formulas can be applied to paper as to other structural materials, such as wood and steel I-beams. Formulas are available that relate tearing strength to fiber length. The same mathematical rules apply to the opacity, or covering power, of paper, as apply to the opacity of paint, films, plastics, and milk glass. On the other hand, it must be admitted that many specifications are strictly empirical or, worse than that, are based on someone's guess or prejudiced opinion. However, progress is being made, and it should be pointed out that when a versatile material such as paper invades a new field, the paper technologist must understand the physical principles and mathematical formulas which apply in that field. The more clearly the papermaker analyzes his problems and the more mathematics he is able to apply, the better chance he has of success in the new field.

Now, while the technical men may often use trigonometry, analytic geometry, and calculus, they are greatly out-numbered by those who use the simpler forms of mathematics, and use them a great deal day after day. I am thinking of the cost accountants, the production schedulers, and the men who take inventory of materials on hand and who forecast coming requirements for materials. There are also those who do market research and try to estimate the probable demands and probable prices for suggested new products. There are the managers—people who have to answer such questions as: "Is this machine obsolete? Shall we replace it with a newer, faster, and more expensive model? Shall we add a new department? Shall we expand and double our production? Can we increase wages? If so, must we increase prices? Where is our break-even point? Where and how can we make a dollar?" Everywhere along the line mathematics is used, and the success of a company depends upon how well and how much it is used.

I should like to give you a few specific examples of how mathematics is useful in the paper industry and, perhaps, to give you a few ideas as to what we'd like to see taught and what we'd like to see emphasized.

Before I became a teacher, I acquired

fifteen years of experience working in paper mills. I worked largely with technical men, but I had ample opportunity to observe the rank and file, and to find out what their foremen, superintendents, and office managers thought of them and expected of them. In the normal run of their work few, if any of them, would be expected to know so complicated a formula as the one for the volume of a cylindrical tank, but any one of them might be expected to read a gauge dial, a chart, a yardstick, or a thermometer and to make a legible record of his reading. Any one of them might be expected to know how to open a valve $2\frac{1}{4}$ turns, or to figure the total weight of $7\frac{1}{2}$ pounds of material plus a $5\frac{1}{2}$ pound container or to know where to stop when told to run $2\frac{1}{2}$ feet out of a tank which originally contained $9\frac{3}{4}$ feet of liquid. These are simple enough tasks but, believe me, they can be important. Relatively few students will ever handle calculus, and when they do they may be dealing only in theory anyway. Many more students will have the opportunity to misread a dial or chart, or put incorrect figures in a record book or run one foot too much out of a tank.

I know that many bosses would like to have men who have a good knowledge of simple arithmetic, whose reading of a gauge or measurement of a stick could be depended upon, and whose night report could be read without guess work the following morning. No teacher of arithmetic or of higher mathematics should ever accept work with numbers so poorly shaped that there can be any question at all as to what they were intended to be. Perhaps there are some people who can never learn to write neatly, but everyone can certainly learn to shape numbers. Legibility should be a natural result of an appreciation of accuracy, and this is one thing which everyone should learn. There is no place in modern industry for sloppy measurements or "bonehead" mistakes in simple arithmetic. As I look back at my mathematics teachers, I have a feeling that they were all too tolerant whenever we had the wrong answers, but "had the right idea," or had "almost the right answer," or had everything right except for the placement of the decimal point, which was one or two places off.

It seems to me that in the grades we should have a lot more practice in actual practical measurements and in deriving correct information from gauges, dials, charts, and tables. With profit to us, we could have been drilled at greater length in the accurate reading of graphs and in the accurate recording of points on various kinds of graph paper. All of this could have been done with a realistic pretense that we were actually engaged in industry and with an explanation of why accuracy is necessary and how much mistakes can cost.

Mention of charts brings to mind the subject of nomographs, these combinations of vertical and slanting scales so arranged that a straight line from a point on one scale to a point on another scale indicates a point on a third scale, which then can be used in further similar operations to derive values which could not otherwise be obtained except by solving complicated formulas. Nomographs are finding increasing use in industry, and a large number of them have been devised for use in papermaking. I believe that somewhere in school students should be introduced to the use of nomographs and that it would be well if more advanced classes had a little practice in making nomographs.

While we are on the subject of the simpler forms of mathematics, I should like to say that in my teaching experience I have been repeatedly surprised at the difficulty experienced in handling simple proportion by some students who have taken calculus. I hasten to add here that such students are at least average in intelligence. Proportion is not a difficult concept, but the students seemingly have trouble recognizing a case of proportion. This is too bad because many of the labo-

ratory calculations in the paper industry are exactly that.

For example, we have a mixture of fibers in water, the concentration of fibers being unknown. We wish to find what volume is necessary to contain 1.8 grams of fiber, so we take a sample of 500 cc. volume and find that it contains 1.23 grams of fiber. Or suppose a 500 cc. sample contains 2.13 grams of fiber and we want to know how much water to add so that 500 cc. will contain 1.8 grams of fiber. Or suppose that paper weighing 20.3 pounds per ream gives a certain tensile test. What would the tensile test be if the paper weighs 20 pounds per ream? Or suppose that a ream of paper weighs 200 pounds when the ream consists of 500 sheets, each $25'' \times 40''$. What would be the weight per 1,000 square feet? Or suppose that X cubic centimeters of a certain dyestuff solution is added to a laboratory sample of fiber. This would be equivalent to how many pounds of dry dyestuff per 1,000 pounds of fiber? Or suppose a 6-inch by 8-inch piece of paper weighed W grams. What would be the weight in pounds of a ream 24 inches by 36 inches containing 480 sheets? Problems such as these, involving simple proportions and conversions from one unit to another, occur frequently. They do not involve complex thinking but the thinking must be clear; the answers must be obtained rapidly and the answers must be correct.

Of course, one must be able to recognize what is *not* a case of simple proportion. Here is a simple algebra problem which has given trouble to many students: How much water should be mixed with a liter of sulphuric acid at specific gravity 1.85 in order to produce specific gravity 1.2? Too often the inclination is merely to divide 1.85 by 1.2 and multiply by 1,000 to get the total diluted volume. That is, the tendency is to confuse specific gravity with concentration in terms of grams per unit volume. This may be a result of careless thinking or of inadequate teaching somewhere along the line. In either case, it can lead to expensive mistakes in manufacture.

For the rather considerable army of routine paper testers in the industry, life is one average after another, and both accuracy and speed are essential. Naturally there is little point in running a series of tests if, when the tests are all done, an error is made in figuring the average and a figure goes into the records which has no relation to the actual quality of the paper. As to speed, there are testing schedules to be maintained. Samples are brought in at regular intervals, often from several machines running side by side, and each sample must be tested for several different properties. Because paper is a variable material, differing slightly from one square inch to the next, every test must be repeated several times. If a paper tester is slow in doing parts of his work, such as in computing averages, he either falls behind or is compelled to make fewer repetitions of each test than would be desirable. Little tricks for speed are therefore good to know, such as multiplying by 2 and moving the decimal rather than dividing by 5, or extracting a dominant number from a column of figures, noticing how each figure differs, plus or minus, from that dominant figure, and finding the average by adjusting the dominant figure by the amount of the average deviations. This sounds complicated, but suppose we have a column containing five 10's, two 9's, and two 11's. Obviously the average is 10. Three 30's and 34 would average 31. In fact, where test values are rather consistent, it is often possible for the paper tester to keep score of the plus and minus values as he goes along and compute the average in his head.

Slide rules are great timesavers, and it should be realized that to do simple multiplication and division on a slide rule, one does not have to be an engineer or know trigonometry or know the mathematical basis of logarithms. In my own experience, I have been pleased to see how quickly the elementary use of the slide rule has been

understood by paper testers whose schooling in mathematics went little further than long division.

Simple averages do not say anything about the variability of tests, so more and more terms like "standard deviation" are creeping into paper work. As you recall, the standard deviation is given by the formula

$$\sigma = \sqrt{\frac{\Sigma d^2}{N-1}}$$

where Σd^2 is the sum of the squares of the deviation of each observation from the mean, and N is the number of observations. Naturally, standard deviation would be a rather difficult thing for the ordinary paper tester to compute with any reliability, but, fortunately, this can be handled by means of a nomograph.

Routine paper testers and their counterparts in other industries are far more numerous than engineers. Hence, it would seem that the mathematical requirements for their work should deserve proportionately more attention from the educational system. It must be recognized, however, that someone is needed to interpret such things as standard deviations and that, in fact, someone has to know enough to ask about them in the first place. In the higher technical echelons of the paper industry, and in the research, development, and control sections of almost every other industry, higher mathematics is a working necessity. In fact, no one can know too much mathematics. Naturally, the kinds of problems handled by these elite few are of special interest to those of you who became mathematics teachers because you liked and wanted to know the subject.

It is the technically-trained few—and there are only a few thousand in the whole country—who worry about such things as correlation, statistical quality control, and underlying mathematical relationships useful in papermaking. It is they who set up the testing schedules for the paper testers. They decide which tests should be run, how often each test should be repeated, and how the sampling should be done. In many cases, they are the men or women who set up the specifications in the first place.

In papermaking there are certain standardized tests, such as the tensile-strength test, the tearing-strength test, the bursting-strength test, the measurement of porosity in terms of rate of air penetration, and, of course, the measurement of weight and thickness. There are also actual use-tests, or tests imitative of actual use. Thus, paper can actually be put into a printing press and printed. It can be formed into cartons, and these cartons filled with merchandise and actually piled into warehouses or shipped around the country to determine how well the paper serves its purpose. It can be actually run through twisting machines that make the cord which is later woven to make paper rugs and seat covers. Careful note can be made of the efficiency of the twisting process and of the quality of the product. Or a mechanism can be arranged in which loaded bags of cement are dropped by a trip mechanism from a certain distance to a concrete floor, and a comparison can be made of how many drops one kind of bag or one kind of bag paper can stand versus another. Of course, the use-tests are the ultimate tests. No matter how well a product survives preliminary tests, its success or failure is measured in actual use.

As a general rule, use-tests are ponderous, slow, and expensive. They are usually destructive to large amounts of material and almost never are they fast enough to serve as control tests. It is better if specifications can be written in terms of the simpler standardized papermaking tests. The problem arises: Which simple test or which combination of simple tests can be substituted for the use-tests? In other words, can a correlation be found between rapid standardized tests and results in the field?

Correlation is a very useful concept for the papermakers, yet many mathematics

students in college have not heard of it. It seems to me that there is a definite value in teaching the simpler aspects of correlation more generally than is now done and at an earlier grade in school. Let me give you an example from my experience.

A manufacturer forms paper plates on large machines, standing perhaps twelve feet high, by subjecting paper to both heat and pressure. He buys paper from many sources and he wants to set up specifications that will assure uniformly high quality in all the plates. He also desires to have the tests run in the individual paper-testing laboratories. In their own plate-making plant, the obvious way to test paper is to make plates out of it and to test the finished plates. But, of course, it is out of the question to install one of the huge platemaking presses in the laboratory of each paper company which might want to sell them paper. Accordingly, an investigation was made in which papers from various manufacturers were tested by various standardized procedures. Plates were made from these same papers and the plates rated as to quality. When the search for a high degree of correlation proved successful, the manufacturer had in his hands a fact basic to the paper plate industry.

Of course, it is not always possible to unearth the actual theoretical basis of a relationship, and sometimes the theoretical equations are too complicated to handle. In such cases it is often possible and useful to develop empirical equations which fit the data fairly closely. While empirical equations are often looked down upon by purists in science, they are highly practical, and the art of developing empirical equations should be taught in the schools.

The use of statistical methods in quality control is comparatively new in any industry and its use is more difficult in the paper industry than it is in many others, but nevertheless it is a coming thing. It has proven its worth in many fields of the paper industry. Nowadays it is hardly possible to pick up a trade journal of the paper industry which does not include either a paper on the application of statistical methods or at least a reference to someone having presented such a paper at one of the numerous technical meetings around the country.

Statistical methods are based upon the mathematics of probability. Raw material variations and other causes create variations in the quality of paper, and machine adjustments are required to compensate for these variations. Statistical methods involve a study of variations. Questions arise, such as: If the finish of a sample of paper selected from a lot is 1.0 low, is the finish for the entire lot off standard? What is the probability of this sample measuring 1 point low even though the entire lot is standard? How many samples and how many tests would be necessary to characterize a whole carload of this paper? Obviously a few scattered tests would not be enough and yet an unnecessarily large number of tests would be wasteful of paper and wasteful of time.

Another problem encountered is: Suppose the tester reports the quality of a sample of paper to be one-half a per cent lower than is specified. Can this variation be tolerated? Should the process be adjusted to correct for it, or does this deviation lie partly in the imperfect precision of the testing instrument itself and partly in certain obscure background variables of the process, variables which come and go without having more than a background effect and often with little, if any, provision for their adjustment?

In answering questions like this last, a control chart is set up for the guidance of the operator or superintendent of the machine. A control chart has three horizontal lines. The middle line represents a design specification, while the other two lines represent control limits or tolerances. As long as the test data fall within the control limits, no machine adjustments are required, and the product will be acceptable because the customer acceptance

tolerances are, as a rule, wider limits than the control limits. On the other hand, if a test datum falls beyond the control limit, this indicates that a correctible deviation has occurred and something must be done.

To illustrate the use of such a control chart, consider a case in which machine tenders were making as many as twenty adjustments during a twenty-four-hour period in order to maintain what they considered standard weight per unit area of paper. The decision to make an adjustment usually depended upon weight measurements of a sample taken at the end of every reel of paper. If the weight for this sample ran a half of a point above standard, the machine was often adjusted to bring the weight back. The adjustment was made without full realization of the variability of the weight measurement. The machine tender needed an answer to these questions: "How much variation in weight in the sample may be expected even though the average weight within the entire reel is standard?" "What is the probability of the weight in the sample averaging a half pound high when the weight in the reel is standard?" In the above case, the machine tender was overcorrecting his process. He raised and lowered his paper weight as a result of sample measurements which were within the normal variation of his process. Each unnecessary adjustment enlarged the variability in the weight of his product.

In this instance, a control chart was set up to aid the machine tenders. They were instructed to make adjustments in weight only when the sample average fell outside the control limits. These control limits were proved satisfactory before such instructions were issued. Since the establishment of this control chart system, it has not been uncommon for the paper machines to operate for twenty-four hours without a weight adjustment. The weight now varies approximately one-half of what it did when over-adjustments by machine tenders greatly inflated the weight variability.

During the war, statistical quality control methods were a "secret weapon," which greatly increased the efficiency and productivity of American industry by holding quality to acceptable standards and at the same time minimizing the amount of testing required. Billions of dollars were saved, and no doubt many lives were saved. Statistical quality control methods are here to stay as a permanent fixture of industry. I believe that teachers of mathematics should recognize this fact and where it is appropriate to do so, they should teach probability and related subjects with that fact in mind.

Of course, when a process is found to be out of control, it is well to know which way to adjust and by how much. A good deal of research is being done to provide this kind of information. As I said earlier, efforts are being made to find underlying mathematical relationships. The paper industry is very much alert to the value of research but progress is slow due to the complexity of the subject. Chemistry, colloid chemistry, many different branches of physics, and physical chemistry are involved. A research man finds use for all the mathematics he can possibly know. In an industry as financially important as the paper industry, the rewards for a really basic advance can be very great, and the industry realizes that its long-term progress must be based on fundamental research. Words like "basic" and "fundamental" are just different ways of saying "underlying mathematical relationships," and I would like to leave with you the idea that the future progress of the paper industry, and for that matter of all industry, depends to a large degree upon how well teachers of mathematics do their job.

A Mathematics Quiz Program

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The following program was developed and staged successfully by one of our senior mathematics classes. One of the boys in the class, acting as chairman, introduced the program as follows:

This morning we are going to have a quite different program—a mathematics quiz. Each of you will have an opportunity to answer all the questions proposed. Let me explain how the program will go. On the stage we are going to present a short scene picturing a typical family. In the course of the action a variety of mathematical situations will occur. Each time instead of giving the answer to the problem the actor will say "BLANK." There will be a suitable pause, timed by Dick, who is standing here; and in the time allowed, you will be able to do your figuring and get your answer on the answer sheets which the boys will hand to you in a few minutes.

Let us have an example to show you how it goes. Suppose on the stage a boy comes in from a trip to the post office and says to his mother, "I got the stamps you wanted—5 airmail stamps at 6 cents each, 30 3-cent stamps and 4 overseas airmail letters at 10 cents each. You gave me two dollars so here is BLANK change." The action of the play stops and you will be expected to figure, "5 airmail stamps at 6 cents each would be 30 cents; 30 3-cent stamps would be 90 cents; 4 overseas letters at 10 cents each would be 40 cents. That makes $1.60, so there would be 40 cents change," and you would write 40 cents in the appropriate space on your answer sheet. At the end of a suitable pause, Dick will say, "time's up," and the action on the stage will continue until another BLANK is reached. Do you see how it goes? Have you any questions?

Each time there is a pause for you to do some figuring, I will hold up a card with the number of the problem on it to help you keep your place. The characters in our play are Mr. and Mrs. Russell, their son Jack, and the maid Nellie.

Let's have the answer sheets passed out now and then we can begin.

The curtain opens on a pleasant living-room with a door to the kitchen-end of the house at the left rear, and a door to the front hall at the right rear. Mrs. Russell stands at the telephone dialing a number. As she dials we hear the sound of the front door opening and closing and she puts down the telephone without completing the connection.

Mrs. R.: Jack, is that you? Where have you been all this time?

Mr. R.: (*He appears in the doorway.*) It's not Jack. I'm home.

Mrs. R.: Oh, it's you. (*Disappointed, she drops into a chair.*)

Mr. R.: Say, Mrs. Russell, is that the way to greet your husband and breadwinner when he gets home all tired out from the day's struggle? What's the matter? (*He, too, sits down.*)

Mrs. R.: I'm sorry, John. I'm so worried about Jack that I can't think of anything else. I'm so glad you're here. Now you can tell me what to do. I was just telephoning the police station when you came in.

Mr. R.: The police station! Why? What has happened?

Mrs. R.: That's just what I'm wondering. Jack left here at 4:15 to go to the grocery for me. It's 5:42 now (*she looks at her wrist watch*), and he isn't back yet.

Mr. R.: Why, Jane, be yourself. You know Jack is a good, dependable boy. He's all right. He'll be right along.

Mrs. R.: But, John, from 4:15 to 5:42 is BLANK minutes. He couldn't have used all that time in just going a couple

of blocks. Something must have happened to him. I tell you, I am going to telephone the police. (*She starts toward the telephone, but Mr. Russell stops her as she starts to pick it up.*)

MR. R.: Hold on! Don't do that. What could have happened to him? He'll be in any minute now and then we'd all be embarrassed if you had called the police. (*She drops into a chair by the telephone table, and Mr. Russell returns to his chair, down left.*)

MRS. R.: Oh, I know you're probably right, but I can't help worrying. Where is he?

MR. R.: Let's say we'll wait until six o'clock before starting out to hunt for him.

MRS. R.: (*She returns reluctantly to her seat, down right, and picks up her knitting.*) Oh, very well. I hope we're doing the right thing. Anything cheerful in the paper?

MR. R.: (*He has brought a paper home with him and now unfolds it.*) No, afraid not. Oh, yes, there is too. I noticed that that radio stock of yours has declared a dividend of $7\frac{1}{2}$ cents a share. That news ought to cheer you.

MRS. R.: It does a little. Let's see. I have 26 shares. That will be BLANK. Well, every little bit helps. (*The front door is opened again.*) Jack, is that you at last?

JACK: Yes, Mom, I'm here. (*He comes in with several packages in his arms, dumps them on the davenport next to his mother, and flops down in the chair near the telephone, center back.*)

MRS. R.: Where have you been? What has happened? Are you all right? What have you been doing? I almost telephoned the police.

JACK: Hey! Give me a chance and I'll tell you.

MRS. R.: Are you all right?

JACK: Sure, I'm all right, but have I had a time! You should have 'phoned the police. You'd have found me, because that's where I was part of the time.

MRS. R.: You were? Why?

MR. R.: Settle down now, son, and tell us your story. What *have* you been doing?

JACK: OK, Dad. It was like this. I started for the store, got there safely, got my things, and was starting back promptly just as you told me to, Mom, when at the corner I had to stop because of an accident. (*Mrs. R. starts to stand up but settles back in her chair as Jack continues.*) Now, hold on, Mom, I wasn't in it. A car was passing me on State Street, when a car tore out from Main Street and side-swiped the one on State. The one on State turned over, but the other one didn't stop at all—just tore on around the corner and down the street. Before you could say 'Jack Robinson,' a crowd collected. They got the people out of the wreck, and laid them on the parkway. Then a police car came up, and soon an ambulance came and took the injured persons—a man and a woman—and then the police began to ask questions of the folks standing around.

MRS. R.: Were the man and woman badly hurt?

MR. R.: They weren't killed, were they?

JACK: They weren't killed. I don't know how badly they were hurt. No one seemed to have noticed the license of the other car and the cop got mad. He said that was always the way, that no one ever kept his head, that usually ninety-six people out of a hundred don't observe a thing that is helpful in a situation like this, and that this crowd was less helpful than most. There were about fifty people in the crowd, so BLANK people should have observed something about the license, and apparently no one in this crowd had noticed it.

MR. R.: You'd think someone would have noticed the license number, but accidents always happen so quickly.

JACK: Uh-huh. Well, Mrs. Russell's little boy stepped up and said, 'I can tell you what that number was.' The policeman asked me why I didn't say so earlier.

'You didn't ask me. I can remember the number because I noticed that the two letters were the first two letters of the alphabet and that there were two pairs of numbers. In the first pair the first digit was twice the second digit. In the second pair, the first digit was the same as the second digit of the first pair, and the last digit was seven. I noticed, too, that the number made by the first pair of digits was four more than the number made by the second pair.' The cop asked me to repeat what I had said. So I said it all over again.

MR. R.: Wait until I get my pencil. (*He hunts around in his pockets.*) Now say it more slowly this time as you did to the cop.

JACK: I noticed that the two letters were the first two letters of the alphabet, and that there were two pairs of numbers. In the first pair the first digit was twice the second digit. In the last pair, the first digit was the same as the second digit of the first pair, and the last digit was seven. I noticed, too, that the number made by the first pair of digits was four more than the number made by the second pair. That made the cop madder than ever and he told me to quit trying to show off my algebra and to give him the number. So I said, 'It was BLANK.'

MRS. R.: You'd think they would have been glad that you were so observant. Why did they take you down to the City Hall?

JACK: Well, the cop said that I'd better tell the whole story to the sergeant. Really they were swell to me, Mom. They brought me home afterward because I had all these packages.

MRS. R.: Oh yes, the groceries. Take them out to Nellie, and then come back and give me the change. (*He picks up the packages and goes off at left.*)

MR. R.: Say, this encourages me to think that the boy is really getting something out of his algebra. I had thought he was pretty slow at it. We'll have to tell Miss Hill about this. (*Jack returns.*)

MRS. R.: Well, how about the change?

JACK: Here it is. The 6 chops were on sale at a special price, 32 cents each. The bread was 15 cents, the celery was 35 cents, and the eggs were 60 cents.

MRS. R.: Say that again.

JACK: The 6 chops were on sale at a special price, 32 cents each, the bread was 15 cents, the celery was 35 cents and the eggs were 60 cents. With the tax, that makes BLANK. You gave me 5 dollars, so here's the change. That's right, isn't it?

MRS. R.: (*She examines the money Jack has given her.*) BLANK, yes, that's right.

JACK: Say, when I went out to the kitchen Nellie was figuring with a pencil and paper. She said that she'd like to ask you a question about some recipe.

MRS. R.: I hate to put my knitting down. Please ask her to come in here.

JACK: (*He goes toward the kitchen door.*) Nellie, mother says to come in here and she'll answer your question. She's afraid she'll drop a stitch if she puts her knitting down. (*Nellie enters with paper and pencil in hand.*)

NELLIE: It's about that pudding recipe you got from the church, Mrs. Russell. Did you know it is for fifty people?

MRS. R.: Why, no. Is it?

NELLIE: Yes'm. It is. You said to make it tomorrow night and I just want to be sure I've got it divided up right. For eight servings I thought it would be all right if I used one-sixth of the recipe.

MRS. R.: Yes, I think that would be all right.

NELLIE: Well, that's easy enough with the eggs because it says to use 24, so I'll use 4, but it says to use two quarts of milk. How much milk should I use? BLANK cups?

MRS. R.: Yes, that's right.

NELLIE: And it says 12 lemons, so I'll use two of those. But now on the butter it says $2\frac{1}{2}$ cups. How much butter shall I use?

MRS. R.: BLANK of a cup.

NELLIE: (*She writes it down.*) Thank you, Mrs. Russell.

MRS. R.: Let me see. (*Nellie steps over and hands her the sheet of paper.*) (*Mrs. Russell examines it a moment, then nods her head and hands it back.*) Yes, that seems all right. (*Nellie leaves.*)

JACK: Say, Dad, Bunny Williams and I are planning to make a miniature of this fort the next time there is a good wet snow that packs. (*He goes to his father and hangs over him showing him a copy of a magazine in which supposedly there is a picture of a fort.*) See, it is 250 feet long. Now if we use a scale of an inch to a foot, that would be 250 inches long, or BLANK feet. That's pretty big, isn't it?

MR. R.: Yes, you'll have to get all of the gang to help you if you make it that big. That's wider than our lot here.

JACK: Well, we'll have to cut down on our scale. Suppose we cut our scale so we use $\frac{1}{2}$ inch to a foot instead of an inch to a foot. Would that mean it would cover half as much ground?

MR. R.: Certainly not! You should know better than that! It would cover BLANK as much ground as before.

JACK: Oh yes, I see. Boy, if Miss Hill were here she'd give us another algebra problem, I bet. It would be like this: If Bunny working alone could build our fort in three hours and I working alone could build it in two hours, how long would it take us working together? Can you do that one, Dad?

MR. R.: Oh, I think so. Let's see . . . Yes, it would be BLANK.

JACK: Right! Now . . .

NELLIE: (*She appears in the doorway.*) Dinner is served.

MRS. R.: Come on, boys, you can finish that discussion later. (*They exit.*)

CURTAIN

The answer sheet had on it a number of entertaining puzzle problems in addition to the spaces arranged for the answers to the problems given in the skit, and at the end of the program a little time was given for the students to work on these puzzles. Then the papers were collected. They were checked by the seniors in their class that day, and the name of the winner in each grade was listed on the school bulletin board.

The success of the program was indicated by the intent faces during the skit and by the many questions asked in each of the mathematics classes that day. Of course, a good deal of careful practice is necessary to make the program go off smoothly.

HAVE YOU READ?

Bowcock, J. E., "Mathematics: In Search of a Soul," *The Mathematical Gazette*, February 1953, p. 7.

This paper was presented by J. E. Bowcock, a student member of the Mathematical Association of Great Britain. He analyzes the place of the mathematics teacher and paints a none too complimentary picture of either our activities or our attitudes. It appears we have not profited by the admonitions given by Dr. Perry over fifty years ago. Although you may and should violently disagree when he says, "Mathematics is the Craftiest Contrivance Satan has yet devised," you must still accept responsibility for his point of view. Only a student believing in mathematics could write such a paper, therefore we can appreciate his remarks.—PHILIP PEAK, Indiana University, Bloomington, Indiana.

Professionalized Subject Matter for Junior High School Mathematics Teachers*

By Myron F. Rosskopf

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Much has been written concerning the training of secondary school mathematics teachers. Leaders in mathematics education have had many words to say about both the educational and mathematical training of prospective teachers. Usually the latter training is divided into pure mathematics and professional mathematics. The concern of this paper is the professional mathematics training of junior high school teachers. Of approximately fifty references that were examined in the preparation of this paper, there were only two that treated explicitly the content of a professionalized subject-matter course.[1] The others made recommendation that such a course ought to be given or that it should be included in a professional-training program but did not give specific information. The question of what to include in a professionalized subject-matter course or how far to go with such a course is still an open question. In this paper an effort will be made to indicate with two or three topics how far to go in such a course. No attempt will be made to indicate what to include in a semester or full-year course.

In order that all of us may have approximately the same concept, let us examine what is meant by professionalized subject matter. In the preface to his textbook, Schaaf writes "Professionalized subject matter may be . . . characterized as that organization of a specific field of knowledge which thoroughly integrates *what* is to be taught with *how* and *why* it is to be taught."[2] He goes on to elaborate this thesis, writing that there must be (1) an "adequate treatment" of the pure subject matter, (2) a presentation of pertinent historical material, (3) a "discussion of educational values and aims," and (4) an indication of teaching procedures. Fehr gives a somewhat similar definition, "Professionalization of subject matter is a scholarly (deep) study of the materials one will teach from the points of view of the subject's foundation, its historical development, its uses, its abstractness and logical structure, and its learning."[3] The difference between the statements of Schaaf and Fehr lies not so much in the words used as in the emphasis given to those words. Fehr puts stress on learning meaning, appreciation, and understanding of subject matter. Schaaf emphasizes teaching methods and educational values of subject matter. It suits the writer's purposes and his point of view to use the following definition: Professionalized subject matter comprises meaning for and understanding of the concepts in mathematics that a teacher will develop with his students; woven into the course is the historical development of the subject matter and appreciation of its place in a mathematical sequence.

A Number System with Base 7

A junior high school teacher is fortunate, indeed, if he has had the experience

* Based on a paper read at the Thirty-first Annual Meeting of the National Council of Teachers of Mathematics, Atlantic City, New Jersey, April 10, 1953.

[1] William L. Schaaf, *A Course for Teachers of Junior High School Mathematics.* New York: Bureau of Publications, Teachers College, Columbia University, 1928.

William L. Schaaf, *Mathematics for Junior High School Teachers, A Professionalized Subject Matter Text.* Richmond, Virginia: Johnson Publishing Co., 1931.

[2] *Ibid.*, p. vii.

[3] Howard F. Fehr, "Professionalization of Subject Matter for Secondary School Mathematics Teachers," from an unpublished paper presented to the members of the Mathematics Institute at Duke University, 1950.

of a contemporary course in the teaching of arithmetic. The reason for such a statement is the result of an analysis of junior high school textbooks that were published during the last ten years; in the case of seventh- and eighth-grade mathematics, approximately 70 per cent of each textbook is devoted to arithmetic. Hence, a professionalized subject-matter course should develop a larger concept of arithmetic and achieve greater depth of understanding of the number system in the participating teachers.

One of the most instructive procedures to use for the purpose of emphasizing the difficulties children have with the algorithms of arithmetic is to construct a number system with a base other than 10 and to work in this number system. Now, suppose 7 is taken as the base of this new number system. What is meant by construction of a number system? How many symbols are necessary? Answers to these questions lead to a thorough understanding of a place-value number system and to a larger concept of the decimal system.

For purposes of computation it is helpful to have an addition table and a multiplication table. These can be arranged conveniently as follows. From the tables it

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

is easy to convince oneself with a few examples that the associative, commutative, and distributive laws hold. For example, the associative law for addition states:

$$6+(5+4)=6+(12)=21 \quad \text{and}$$
$$(6+5)+4=(14)+4=21:$$

also, the distributive law gives:

$$2(3+2)=2(5)=13 \quad \text{and}$$
$$2(3)+2(2)=6+4=13.$$

Now, let us examine "carrying" in an addition problem.

Write,

$64 = 60 + 4 =$ 6 sevens + 4 units
$25 = 20 + 5 =$ 2 sevens + 5 units

Adding,

$110 +$ 12 = 11 sevens + 12 units

Simplifying,

$122 =$ 1 forty-nine + 2 sevens + 2 units

Use was made of the addition table to secure the partial sums $4+5$ and $60+20$. Now the exercise can be done mentally as follows: 5 and 4 are 12; write the 2 and carry the 1 seven; 6 sevens and 1 seven are 10 sevens; 10 sevens and 2 sevens are 12 sevens; hence the sum is 122. With a little practice one can become quite adept in addition in this number system.*

* At this point in the presentation there occurred an exchange of remarks among some of the participants in the meeting concerning the referent of the term "seven." After the meeting, Gertrude Hendrix, Eastern Illinois State College, Charleston, Illinois, wrote me a long note concerning the confusion. With her permission I wish to present the following summary of her remarks. "[One] thinks that the word 'seven' refers to the symbol '7.' Hence, if [one does not] have the symbol '7' anymore, as in writing numbers to the base seven, then he does not have anything for the word 'seven' to refer to. That is, [one seems to think that] numbers *are* numerals! The case is one of the most perfect examples of confusion of symbol with referent that I have ever found. Not that the [questioner] does not possess the concept to which 'seven' refers in the natural number system; he would pass all behavioral tests of knowing that 'seven' is the property of all groups of discrete objects or events whose members can be arranged in

Next, consider a multiplication exercise. You will find that the rationale of the algorithm for the base 7 is exactly the same as that for the base 10. For example:

$$\begin{array}{rl} 325 & (4+60)325 = 4\cdot 325 + 60\cdot 325 \\ 64 & \\ \hline 1636 & \\ 2622 & = 1636 + 26220 \\ \hline 31156 & = 31156 \end{array}$$

Moving the second partial product to the left one seven-place is due to multiplication by 6 sevens; this is the same situation as in the base 10. Only in the latter case a partial product is moved one *decimal* place to the left because of multiplication by a certain number of tens.

Subtraction of a smaller number from a larger number can be done by a process of "exchanging." In the example $612-35$, you cannot subtract 5 units from 2 units. Hence, you exchange 1 seven for 10 units and subtract 5 units from 12 units; the result, as seen from the addition table, is 4 units, because $5+4=12$. To continue the exercise, you cannot subtract 3 sevens from 0 sevens; so exchange 1 forty-nine for 10 sevens. Then 3 from 10 is 4, by the addition table. Hence, the result of the subtraction, or the difference, is 544.

Doing several exercises in this way will enable teachers to acquire both meaning and understanding for subtraction. Mastering subtraction in this strange number system will make these teachers better able to help those junior high school children who need remedial teaching. Moreover, the teachers are forced to use the additive definition of subtraction on an operational level. They may or may not state explicitly this definition, but they are accustomed to operating with it. Later on the definition can be given explicitly.

Long division is difficult to learn, even in the decimal system. Let us see if long division is any easier in the base 7 number system. In the division $6024\div 23$, first, estimate how many groups of 23 will be contained in 6024; there will be between 200 and 300 groups of 23, since 23×200 is 4600 and 23×300 is 10200. Hence, the number of *forty-nines* in $6024\div 23$ is 2; that is, our first estimate of $6024\div 23$ is 200. Subtract this group of 23's, arranging your work as follows. The difference is 1124; estimate the number of groups of 23 that is contained in 1124. There will be between 30 and 40 groups, since $23\times 40=1250$. The work now can be arranged as indicated. Proceeding in a similar fashion to the foregoing discussion, estimating and calculating, you can complete the long division to obtain the quotient of 233 and the remainder of 2. After only a few exercises with teachers the "useless" zeros can be eliminated and the usual long division algorithm obtained.

$$\begin{array}{r} 200 \\ 23\overline{)6024} \\ 4600 \\ \hline 1124 \end{array}$$

$$\begin{array}{r} 3 \\ 30 \\ 200 \\ 23\overline{)6024} \\ 4600 \\ \hline 1124 \\ 1020 \\ \hline 104 \\ 102 \\ \hline 2 \end{array}$$

One could go on with the development of a base 7 number system, introducing positive and negative numbers, fractions, and decimal fractions. However, the work with zero and the natural numbers is usually sufficient to develop an appreciation of the power and the efficiency of a place-value number system. Instead of pursuing further operations in the base 7 number

one-to-one correspondence with the members of this group of marks: : :. He is fully *aware* of seven as a property of all such classes of individual objects (or events)—that is, as the selector or identifying property of a class of classes. But this afternoon when *we* were talking about the property of classes of individuals, *he* was thinking (and talking) about the numeral '7'."

Semantic difficulties arise often in classes. It is necessary that teachers themselves use language correctly, otherwise language becomes a barrier to communication that will interfere with the learning of mathematics. Teachers need to have meaning for concepts in mathematics in order that they may develop with their students these concepts to an extent appropriate to the maturity level of the students. It cannot be over-emphasized that all of us engaged in teaching must clearly distinguish between symbol and referent.

system, let us formulate a finite number system.

A Modular Number System

The finite number system that will be discussed consists of the five symbols 0, 1, 2, 3, and 4 and two operations, + and ×, defined by the following tables.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Such a number system is *closed* with respect to the operations + and ×; that is, two numbers combined with the operation + or the operation × is a number of the system. This is usually called the *law of closure*. By consideration of specific examples, you can convince yourself that the associative, commutative, and distributive laws hold. Such examples follow:

Associative Law

$$2+(3+4)=(2+3)+4$$
$$2+(2)=(0)+4$$
$$4=4$$
$$2\times(3\times4)=(2\times3)\times4$$
$$2\times(2)=(1)\times4$$
$$4=4$$

Commutative Law

$$3+4=4+3 \qquad 3\times4=4\times3$$
$$2=2 \qquad 2=2$$

Distributive Law

$$2\times(3+4)=(2\times3)+(2\times4)$$
$$2\times(2)=1+3$$
$$4=4$$

The difference between two numbers a and b of the system is a number x, providing it exists, such that $a=b+x$. The table of differences can be arranged as follows if the subtraction is interpreted as $a-b$ and the value of x is given at the intersection of the ath row and bth column. You observe no blank spaces in the table; hence, the value of x always exists and subtraction always has a meaning. As you would expect, subtraction is not associative nor commutative. For example, $4-(2-3)\neq(4-2)-3$; the value of the left member is 0, while the value of the right member is 4. Observation in the table shows that $2-4$, say, is different from $4-2$.

a \ b	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Division is defined in terms of multiplication. The quotient $a\div b$ of two numbers in the system is a number x, providing it exists, such that $a=bx$. With this definition of division the adjoining table can be constructed. It is observed that a quotient exists in all cases except when the divisor is zero. As in the case of the subtraction table, the quotient is always $a\div b$ and x is given as the intersection of the ath row and bth column.

a \ b	0	1	2	3	4
0	–	0	0	0	0
1	–	1	3	2	4
2	–	2	1	4	3
3	–	3	4	1	2
4	–	4	2	3	1

Number systems such as the one we have constructed are called *modular* systems, and the number of elements in the system is called the *modulus*. In our case, the modulus is 5. The addition and multiplication tables are constructed in the same way; in the case of the sum of two numbers or the product of two numbers the result is taken as the smallest remainder after division by the modulus. That is, $4\times4=1$, because $(4\times4)\div5$ has a smallest remainder of 1. Sometimes a modular number system is called "arithmetic on a dial" or "algebra on a dial" because of the similarity between the constructed number system and a clock face. A clock, of course, is an example of a modular system in which the modulus is 12.

In the solution of a quadratic equation by factoring use is made of the theorem in our ordinary number system that $a=0$ or

$b=0$ if $ab=0$. The same theorem holds in our finite number system. An examination of the multiplication table shows that the only way a zero product can be obtained is to have one factor of the product equal to zero. The zeros are along the borders of the table; no zeros appear anywhere else.

Now, this is a property peculiar to some finite number systems of which the one we have constructed is an example. So long as the modulus of a number system is a prime number, the theorem regarding $ab=0$ holds. If, however, the modulus is not a prime number, the theorem no longer is valid. Consider the multiplication table for a system in which the modulus is 6. Notice that zeros appear in the body of the table. It is possible to have a product equal to zero, and neither factor is zero. For example, 2×3 and 3×4 are zero products. Such elements in a system, like 2, 3, and 4 in the modulus 6 system, are called *divisors of zero*. This means that a quadratic equation does not admit of unique solutions in a modulus 6 number system. It is rather remarkable that our complex number system, which grew as need arose, is such that you can say all polynomial equations of positive degree with complex coefficients admit a solution in the system of complex numbers.

x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Positive and Negative Numbers

As pointed out in a series of papers[4] appearing in The Mathematics Teacher, you have a choice of procedures for extending the system of natural numbers to a system of positive and negative numbers including zero. You can take the position of assuming an *integral domain* made up of zero and the positive and negative numbers; then, on the basis of the assumption that the associative, commutative, and distributive laws hold, prove the laws of signs for operating with positive and negative numbers. Or, you can take the position of introducing positive and negative numbers by proper definition (couples of natural numbers, for example) and define the laws of signs for operating with positive and negative numbers; then you prove that the associative, commutative, and distributive laws hold for these numbers.

In preparation for this paper, the writer reviewed his many years of day-to-day classroom experiences with secondary school students. On the basis of what he did in a classroom and what he has learned since leaving a secondary school classroom, the following steps for developing a concept of positive and negative numbers are recommended.

Step 1. The first step consists of a long period of gradual introduction of students to the concept of a negative number. Some of the foundation for a concept of a negative number is laid in the elementary school. In connection with instruction in reading a thermometer, some teachers connect a below zero or below freezing temperature with a negative number. As a student progresses through school, his concept of a negative number should be extended to include other situations. In addition, positive numbers should be introduced to contrast with negative numbers. You are leading here to the concept that $+3$ and -3, for instance, are opposites. During the eighth grade or the ninth grade there should be an intensification of the process of introducing occasions for interpreting a situation in terms of positive and negative numbers. The important point to stress is that these experiences of students should be spread over a long

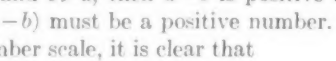
[4] Henry Van Engen, "Logical Approaches to $(-a)(-b)=ab$ and $x^0=1$," The Mathematics Teacher, XL (April 1947), 182–185. Howard F. Fehr, "Operations in the Systems of Positive and Negative Numbers and Zero," The Mathematics Teacher, XLII (April 1949), 171–176. Henry Van Engen, "On Understanding Mathematical Methods," The Mathematics Teacher, XLIII (December 1950), 397–402.

period of time, the first experience being isolated instances and the later experiences being planned so that there is an increasing concentration of them.

Step 2. In this step there is a shift from a multiplicity of interpretations for positive and negative numbers to a geometric interpretation of them, namely, the number scale. On an intuitive level, develop with students the process of addition of positive and negative numbers, using the number scale. Lead them to the point where they are ready to make a generalization themselves. A word statement of the generalization for adding positive and negative numbers is desirable but do not insist upon every student being able to have meaning for the statement of the generalization. Be content if every student can work correctly with addition.

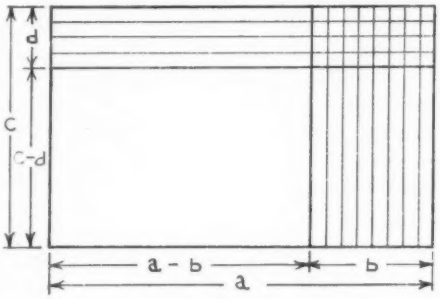
Step 3. By means of specific examples using natural numbers build up a background for the explicit statement of a definition of subtraction in terms of addition. Use a number scale and the definition of subtraction agreed upon to subtract positive and negative numbers. With the class do a large number of examples; try to lead the group to formulating the statement of a general "rule" for subtracting positive and negative numbers. If you find a particular class is not ready for the statement of a general rule, continue to work with the number scale and the definition of subtraction. Keep the work basic for meaning until a class is ready for a higher level of abstraction.

At this point, you may wish to introduce a simple rationale for removal of parentheses. If a, b, and c are positive, $a>b$ and $c>a$, then $a-b$ is positive and $c-(a-b)$ must be a positive number. On a number scale, it is clear that



$a-b$ is represented by the line segment between point a and point b; also $c-(a-b)$ is the line segment that remains after $a-b$ is taken away from c. This line segment (in two parts) can be obtained by taking a away from c and adding b to the difference. That is, we have $c-(a-b)=c-a+b$.

Step 4. In this step use will be made of what Klein calls a " . . . general peculiarity of human nature . . . , by virtue of which [one] *involuntarily* [is] *inclined to employ rules under circumstances more general than are warranted by the special cases under which the rules were derived and have validity.*"[5] The time has come to develop a rationale for the definition of rules for multiplying positive and negative numbers. Suppose a, b, c, and d are four positive numbers such that $a>b$ and $c>d$. Then $a-b$ and $c-d$ are positive numbers.



Draw a rectangle with length a and height c. On the length mark off b and draw a line parallel to the height. On the height c mark off d and draw a line parallel to the length. Then the area of the blank rectangle is $(a-b)(c-d)$, but the area of this rectangle can be computed by subtracting from the area of the largest rectangle the areas of smaller rectangles. That is:

$$(a-b)(c-d)=ac-ad-bc+bd$$

where bd must be added because the double cross-hatched rectangle was subtracted twice in subtracting the area of

[5] Felix Klein, *Elementary Mathematics From an Advanced Standpoint*, pp. 25–27. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York: The Macmillan Company, 1932. The essence of the development in this section is in the spirit of the teaching suggestions Klein makes concerning positive and negative numbers.

the rectangle with horizontal lines and the area of the rectangle with vertical lines.

It is at this point that use is made of the tendency of human beings to extrapolate commented on by Klein. The foregoing formula has been developed only under the conditions that $a>b$ and $c>d$. Now, if the formula were true when $a=c=0$, you would have $(-b)(-d)=bd$, the rule for multiplying two negative numbers. By use of rectangles and this frailty of human nature, you could develop the rule for multiplication of a positive and a negative number. The end result of the rationale is a definition for multiplication of two positive and negative numbers.

Summary

In this paper a definition of professionalized subject matter was stated in terms of meaning and understanding of mathematical concepts that a teacher will develop with students, the historical treatment of this subject matter, and the place of the subject matter in sequential mathematics. We went on to outline a presentation for assuring meaning and understanding for operations (1) in a system of natural numbers with base 7, (2) in a finite number system, and (3) with positive and negative numbers. Because of limitations of space and time the historical treatment of man's concept of number and operations with number was not touched upon.

We have been explicit in the delineation of these subject matter topics because there exists no detailed analysis in the literature. There remains to be spelled out a treatment of other topics that are properly part of a professionalized subject matter course for junior high school mathematics teachers. Then, by pooling the efforts of leaders in mathematics education, a person engaged in training prospective junior high school mathematics teachers can devise a semester or a year course that will result in teachers who will be better equipped to motivate and to present mathematics so as to achieve for students concepts that are correct, have meaning, and are understood.

HAVE YOU READ?

Dalton, A. G., "The Practice of Quality Control," *Scientific American*, March 1953, p. 29.

What are the chances that your next boiled egg will be the way you like it? If the statistician was preparing it, he would use "Quality Control" to assure you an egg cooked to suit your sensitive palate. Quality Control has been surrounded for a number of years with a mystic veil of statistical jargon. This article lifts that veil and allows the nontechnical reader to realize the value and understand the applications of this important branch of statistics. Quality Control aids in (1) giving warning of abnormal behavior, (2) diagnosing the underlying courses of wastefulness, and (3) helping to establish economical inspection procedures. This article gives an excellent description of an important use of statistics.

Faster, James E., "Don't Call It a Science," *Mathematics Magazine*, March–April 1953, p. 209.

Do your students ever ask you for a definition of mathematics? Have you ever quoted Bertrand Russell's famous definition to them? This article presents a stand on the classification of mathematics. Is mathematics a domain of abstract implication? If so why can't it be classified with the sciences? Does mathematics and science put the same emphasis on observations? How do mathematics and science compare in their use of the piling up of evidence to be used in accepting a hypothesis? For example, how would the proof of the Pythagorean theorem differ when done by the Mathematical Method *vs.* the Scientific Method. Here is an article you will not be indifferent to.—Philip Peak

The Principle of Linearity—Theory and Application

By JOHN W. CELL
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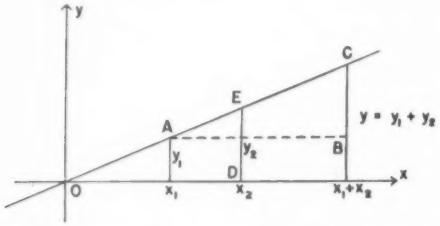
WE BEGIN by proving a very simple theorem. Consider a straight line which goes through the origin in the xy-plane as shown in the figure.



THEOREM: If y_1 is the ordinate to a straight line which goes through the origin at abscissa x_1 and if y_2 is the ordinate corresponding to abscissa x_2, then the ordinate corresponding to abscissa x_1+x_2 is y_1+y_2.

For the proof we note that triangles ODE and ABC are congruent and the proof is immediate. We proceed to utilize this theorem in several situations.

Hypothesis: The graph of $y=1/x$ is a straight line which goes through the origin.

Conclusion: $1/x_1+1/x_2=1/(x_1+x_2)$. The proof is immediate.

Hypothesis: The graph of $y=\sqrt{x}$ is a straight line which goes through the origin.

Conclusion: $\sqrt{x_1+x_2}=\sqrt{x_1}+\sqrt{x_2}$. Again the proof of this algebraic result is immediate from the basic theorem. We note in passing that clearly

$$\sqrt{9+16}=\sqrt{9}+\sqrt{16}.$$

Hypothesis: The graph of $y=\log x$ is a straight line which goes through the origin.

Conclusion: $\log(x_1+x_2)=\log x_1+\log x_2$. The proof of this relationship follows immediately from the basic theorem for the straight line. We observe in passing that a great many college freshmen have little difficulty in learning this rule and in using it at every opportunity.

Hypothesis: The graph of $y=\sin x$ is a straight line which goes through the origin.

Conclusion: $\sin(x_1+x_2)=\sin x_1+\sin x_2$. The proof of this theorem follows at once from the basic theorem. This theorem, like the preceding, is typical of those which the freshman has little difficulty in learning and in utilizing whenever an opportunity arises.

The reader will long since have realized that the writer has his tongue in his cheek with respect to these so-called theorems. The fallacy, of course, is in the hypothesis in each case—none of the graphs is a straight line! However, the resulting algebraic forms are met so often on freshman papers that they and like mistakes constitute at least a half of the mistakes which are made.

Could one reason for the prevalence of this type of mistake be that too little emphasis is placed in high school and in the freshman year in college on the rules of manipulation of equations and on the ALGEBRA OF FUNCTIONS. The derivation of the basic rules for each new function and a thorough understanding are fundamental.

Now that we have discussed the false principle of linearity let us turn to the correct principle. Consider the very simple algebraic equation:

$$ax=b+c, \qquad a\neq 0.$$

It is at once evident that the solution of this one equation is the algebraic sum of the solutions of the two equations:

$$ax=b \quad \text{and} \quad ax=c.$$

A very elementary application of this fact can be made to solve the equation $15x=243.6$ for the root in decimal form. Thus, we could solve $15x=240$ and $15x=3.6$, and add the results.

This equation is an example of a so-called simple equation or of a linear equation in one unknown.

Let us consider n simultaneous *linear* equations in n unknowns and let us suppose that the determinant of the coefficients is not zero so that a unique solution does exist. (We illustrate with three equations in three variables.)

$$a_1x+b_1y+c_1z=d_1,$$
$$a_2x+b_2y+c_2z=d_2,$$
$$a_3x+b_3y+c_3z=d_3.$$

Now by the usual rules, the solution for x is given by

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Similar expressions give the solutions for y and z. Let D denote the denominator of the preceding. Then

$$D\cdot x=d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+d_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and, evidently,

$$D\cdot x=d_1\begin{vmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}+d_2\begin{vmatrix} 0 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}+d_3\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix}.$$

The forms for the y and z parts of the solution can be treated in the same manner. We have thus established the very basic theorem:

THEOREM: Let the solutions of the three simultaneous equations with right-hand members 1,0,0; 0,1,0; and 0,0,1 be respectively x_1,y_1,z_1; x_2,y_2,z_2; and x_3,y_3,z_3. Then the solution of the given three equations is $x=d_1x_1+d_2x_2+d_3x_3$ and similarly for y and z.

Note that the proof of this theorem is easy by the rules of determinants. On the other hand if we wanted to solve a set of such equations with numerical coefficients, the method of determinants would be much too long while the methods of addition-subtraction or substitution would be more appropriate. Besides, the method of addition-subtraction, with a slight addition, can be checked at every stage of the work (the Doolittle method).

This theorem finds ready application in civil engineering in the consideration of a truss with many loads. It is quite common to see engineers solve for deflections due to a one-pound load at one of the positions and zero loads at the other, and then to combine to get the total situation. There are good design reasons for proceeding in this manner.

Electrical engineers use this fundamental principle repeatedly as they solve a multi-mesh electric circuit for the currents due to voltages throughout the meshes. Other engineers and scientists use this principle and term it the principle of superposition.

We could follow this same linearity principle through calculus (e.g., the derivative of a sum is the sum of the derivatives; the integral of a sum is the sum of the integrals).

Consider next the differential equation:

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4x+5e^{2x}.$$

This, of course, is an example of a *linear* differential equation with constant coefficients. To obtain the general solution we may find the so-called complementary solution which is the solution of the equation with zero for the right-hand member, we may find a particular solution of the equation with only $4x$ for the right-hand member, and we may find a particular

solution with $5e^{2x}$ as the right-hand member. We then add algebraically the three to obtain the general solution. Note the same principle of superposition applying here again.

Engineers use this idea repeatedly in solving for effects due to a number of different causes. It is a correct method when the basic mathematical differential equation is *linear*. And the same general type of idea applies to other mathematical equations.

It is worth digressing to observe that the student of mathematics who really knows that **if equals be added to equals, the results are equal,** and similarly for subtraction, for multiplication, and for division by any constant not zero, will find repeated chances to use these ideas as he attacks each new type of equation. Nevertheless in the following equation it is difficult for many sophomores to solve for the integral

$$\int e^{2t} \sin 3t dt = -\frac{1}{3} e^{2t} \cos 3t + \frac{2}{3}\left[\frac{1}{3} e^{2t} \sin 3t - \frac{2}{3}\int e^{2t} \sin 3t dt\right].$$

What is the situation when we consider any non-linear equation? How about the general solution of a high degree equation in one variable? How about the simultaneous solution of two quite general quadratic equations in two variables? For example, how would you suggest that we solve simultaneously the two following equations:

$$2x^2+3xy+4y^2+5x+6y+7=0,$$
$$8x^2-7xy-6y^2-3x-2y-11=0?$$

Does the principle of superposition apply? Thus, can we solve the above two equations with constant terms 1 and 0 and again with 0 and 1, and then combine to obtain the required solution of the given problem? The answer is clearly in the negative.

Probably many of you would solve these two equations simultaneously by graph and this would be an excellent method. Others might eliminate by substitution, simplify to obtain a fourth-degree equation in one unknown, solve this by some approximate method to the required accuracy, etc. But any method used will be tedious.

The same situation persists when we consider non-linear differential equations. Yet some phases of science need desperately to know a great deal about the solutions of particular non-linear differential equations. Consequently, the modern high-speed computing machines have been developed as one mode of attack. However, their bottleneck is not the non-linearity but rests in other facets of this type of problem.

In this paper we have considered the false principle of superposition and the correct application and proof. There are other basic facts that apply to linear equations and which are utilized frequently in engineering and science. We have digressed at points to observe how sound early training stands in good stead in subsequent mathematical problems.

"There are in science immense numbers of different methods, appropriate to different classes of problems; but over and above them all, there is something not easily definable, which may be called the method of science. It was formerly customary to identify this with the inductive method, and to associate it with the name of Bacon. But the true inductive method was not discovered by Bacon, and the true method of science is something which includes deduction as much as induction, logic and mathematics as much as botany and geology."

—Bertrand Russell, *Mysticism and Logic* (1918).

On Asking Questions

By T. M. Simpson, *Professor Emeritus of Mathematics*
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A teacher of college mathematics learns after some experience with freshmen that it is more important to ask a good question than to answer it. The writer has been experimenting with a class in analytic geometry, trying to direct their thinking into a clearer realization of the usefulness of self-put questions. Repeatedly an instructor will ask a question of the class and receive an answer unrelated to the question, or one which indicates that the nature of the problem did not register. Communication between the teacher and the class breaks down. He asks for an equation and gets a formula. He inquires about factors and gets roots. Seeking the area of a triangle, he obtains a guess at the vertices. Examples can be multiplied *ad infinitum*.

If the student can be taught to state a given problem in several different ways, after awhile the solution may appear. It is not easy to get students to state a problem a way different from the way it is written, but persistent questioning by the instructor will usually produce results. The point I am making is for the instructor not to plunge into the solution of a problem easy for him, before the problem has been thoroughly stated in a variety of ways. Putting the question is the important thing. Indeed, when this has been done carefully, it frequently happens that the student's curiosity is aroused and we have solved the important problem of motivation. Of course, one great difficulty is lack of vocabulary and not knowing the meaning of words. Definitions are particularly troublesome. The student can memorize a definition, but it may be nothing but words without meaning for him.

Here is an example of a struggle over restating a problem. *Example:* A point moves so that its distance from the point (3,2) is proportional to its distance from the line $x+y-10=0$. Find the equation of the locus described. This is hard for freshmen. There are numerous ideas—distance from a point to a point and to a line, proportional to, locus, etc. So I begin with trying to extract a few preliminary comprehensions. "What does 'y is proportional to x' mean?" Nobody knows.

One ventures finally, "As one goes up the other does."

"No!"

"It is in the same ratio."

"Well, what does that mean?" Silence.

Then, "One changes as much as the other does."

"No."

Another hopefully declares, "It means they are equal."

Finally, it is dragged out that the meaning is $y=kx$. Then I ask the class to say it another way, several other ways. For example, the fraction y/x always has the same value. The ratio of y to x is constant. Is the ratio of x to y a constant? What if y and x are multiplied by the same number? This takes time. Now take the original problem and say it another way. After this is "batted" around awhile it may be possible to get somebody to word it, "The distance from the point (x,y) to the point (3,2) equals k times the distance from the point (x,y) to the line $x+y-10=0$." But first, of course, it must be discovered that the point which moves is named (x,y). When the problem has finally been worded so that the class comprehends it, the solution itself is not so difficult.

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It does no harm to ask the class what we are going to have when we get the answer. If the problem requires the equation of a line then the answer is an equation. The x-intercept for a curve is a number, and an answer like (3,0) is not correct. The sin of 30 degrees is $\frac{1}{2}$. ($\frac{1}{2}$ of what?). Such questions may seem trivial, but for the struggling student they are not. For example, what is the *relation* between x and y if the distance between the points (3,5) and (x,y) is 8? A relation to some means a ratio. Does the class know that relation here means an equation containing the unknowns x and y?

Nobody knows better than a mathematics teacher the importance of clear, intelligible statements in which each word as well as the total is meaningful to the student. The textbook will say, "Express y in terms of x." The student is asked to say this another way and generally he doesn't know any other way; some students do not know what it means. The text records, "For some purposes it is desirable to express the cartesian coordinates in a form involving trigonometric functions." The writer may as well omit that sentence as far as the student is concerned. It gives the student a fuzzy feeling in his head if he reads it. Another expression that throws the class back on its heels is this, "When x increases continuously from 0 to infinity. . . ." Try putting a panel to work on letting x increase continuously and see what happens.

It has been a standard joke in referring to a sermon that no souls are saved after the first twenty minutes. The joke, of course, involves a serious truth—that the mind will not attend to an organized discourse very long. And yet teachers will talk for forty-five minutes on an involved topic in mathematics and expect the class to follow the thought. Frequently the teacher will ask, "Any questions?" When none are forthcoming, he will assume that the class understands. Generally, he should assume the contrary. Better results might be obtained if the teacher talks no more than ten minutes, and instead of assigning problems, assigns the task of asking questions, or of restating problems without solving. This is harder on the teacher, but it may make the class do more thinking, and prevent them from "looking for a formula," or working back from the answer.

Any teacher who varies his usual class procedure by trying this out will get interested and the class will have a challenge. The idea is bound to come up, either expressed or implied, that "there is no other way to put the problem." Yet there always is. For example: Extract the square root of 1201. Alternative statements:

(1) What number multiplied by itself will produce 1201?
(2) The square of what number is 1201?
(3) If 1201 is resolved into two equal factors, find one of them.
(4) 1201 is divided by a number x and the quotient is x. Find x.
(5) A certain two-digit number (why two-digit?) when multiplied by itself gives 1201 as the product. If the number is $10a+b$, how can we find a and b, and so the square root of 1201? This formulation, of course, leads to the classic way of extracting the square root.
(6) Referring to (3) above, if we *guess* one factor to be 30, say, divide 1201 by 30, take the average of the divisor and dividend, divide 1201 by this average and keep repeating the process, will this lead to two equal factors of 1201?

Other examples readily come to mind. In this connection, too, the student should have it called to his attention that many of the great discoveries in science were made by people who began by asking an old question in a fresh way, or by asking a new question, suggested by rephrasing an old one. After two thousand years of Euclid's parallel postulate and many attempts to prove it, a man of genius asked himself a simple question, is it true? and so invented non-Euclidean geometry.

Statistical Training for Secondary Schools*

By William J. Moonan

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1. The Nature of Statistics

Before we embark upon a discussion of the role of *statistics* and the *statistical method* in the curriculum of secondary schools, it is necessary that we have a mutual understanding of certain terms and concepts. In view of the fact that "statistics" means many things to many people, a definition will be proposed so that a basis for discussion may be shared.

In the broadest sense, "statistics" provides techniques for the collection, analysis, and interpretation of data which are acquired for scientific purposes. This definition is quite satisfactory for our requirements except that it pays little heed to the role played by the theory of probability in developing the techniques. By means of courses in mathematical statistics the role is made quite explicit. In these courses, the mathematical development of the statistical technique is accomplished. The objectives of more practical courses in statistical methods are to show the utilitarian aspects of these techniques rather than their derivation. It is here that the relationship of statistics to the *scientific method* is brought into sharp focus. Let us review the components of the scientific method so that we may see exactly where statistics enters in. The scientific method involves the following:

1. A statement of the problem
2. A statement of an hypothesis
3. The design of an experiment to test the hypothesis
4. The collection of data by means of an experiment
5. The analysis of the data
6. A conclusion is formed regarding the hypothesis
7. Action is taken or recommendations are made.

It can be shown, for particular problems, that statistics enters directly into the procedures of Steps 2, 3, 4, 5, 6 and indirectly into Steps 1, 7. The scientific method has been practiced long before modern statistical methods were developed, but the point is that current statistical practices make the collection of data more objective, the analysis of data more precise, and the interpretation of data more appropriate. Those who are willing to submit their opinions and problems to the cold-blooded scrutiny afforded by the scientific method will find statistics necessary and valuable.

The fifth step of the scientific method concerns the analysis of the data. This analysis refers to the calculation of the appropriate numbers (called statistics) from the data that summarize the information of the data. The data themselves are frequently presented in the form of tables and charts. The common concept of statistics is that it refers to the methods used to tabulate the data and present the graphs and charts. In reality, these activities are often relegated to competent clerks by statisticians who realize that such abilities are important and should be a part of the basic training of elementary students. Tabulating, graphing, and summarizing data by means of measures of central tendency and variability are called *descriptive statistics*. The procedures and techniques used in testing hypotheses and making inferences from data are called *inferential statistics*. There are some statisticians, myself included, who believe this is a false dichotomy. They maintain that the place for the consideration of descriptive statistics is in the course of their

* Part of a lecture given to the Spring Conference of Mathematics Teachers of Minnesota, at Coffman Memorial Union, University of Minnesota, Minneapolis, April 25, 1953.

natural need when performing the acts of inference.

When addressing a group of mathematics teachers, it seems advisable to state the relationship between mathematics and statistics. It may come as a surprise to some that they are not identical. The difference between them may be exemplified by stating that mathematics is an exact science, whereas statistics attempts to evaluate scientific endeavor. These attempts result in probabilistic statements, equations, or values, whereas the results of mathematical analysis end in logical statements, equations, or values. It is true that statistical theory is dependent upon mathematics, but statistical methods deal with the realities of life and not abstractions. No statistician would deny that the more mathematics you know, the better off you are in statistics, but many able statisticians would deny being mathematicians. There have been several individuals who have made significant contributions to our knowledge in both mathematics and statistics. Two such men are Ronald A. Fisher and the late Abraham Wald. The true genius of these men and the heritage they have provided are not generally appreciated or understood.

As a final remark in this section, it is important to point out that the results of modern physics show that the nature of the universe is not mathematical, but statistical. Two specific illustrations of the reality and importance of this conclusion are the Heisenberg Principle of Indeterminancy and the inclusion in the physics curriculum of a course called "Statistical mechanics." The current theories of heredity and evolution also have statistical bases.

2. Pros and Cons of Teaching Statistics in Secondary Schools

In this section we shall briefly consider some points relevant to teaching statistics to high school students. It can hardly be expected that a statistician would give a convincing argument against teaching statistics to anyone. This is true not only because he owes certain allegiances to his profession but also because he believes that the subject has important implications for rational living.

Some competent authorities have maintained that secondary pupils are too immature and lack the mathematical sophistication needed to profit very much from a course in "statistics." I would heartily agree to this idea if by a "statistics" course is meant "descriptive statistics" as commonly given as an introductory course in universities and colleges. The material therein is usually concerned with the evaluation of means, medians, modes, ranges, percentiles, standard deviations, and correlation coefficients. Such information is quite suitable for students entering a certain profession and who have the procedures and data, which are used as examples, oriented to subject matter they are studying. However, the traditional approach to descriptive statistics is anything but appealing. It is true that elementary students can learn such material well enough, but the point I wish to make is that it is not realistic. It is here that I depart from some of my respected colleagues who believe that courses in descriptive statistics are justifiable for either college or high school students. I can only reconcile myself to believe this if the presentation of the statistics is made naturally by the necessity for the student to study practical and important problems. We gain very little in life through description of the data of our environment, but we profit considerably if we can make reliable and valid inferences from it and plan to do so before the data are collected. For this reason I believe more emphasis should be placed on inferential and less on descriptive statistics in both college and high school. I am afraid that if we leave high school students (as we do many college students who do not take more work in statistics) with only the knowledge and abilities acquired by learning descriptive

statistics as it is usually taught, we have done them great disservice. This is true because they are not provided with the means whereby they can test hypotheses from data. Of course, elaborate statistical methods cannot be introduced, but some fundamental concepts may be acquired and useful techniques learned.

Another objection to teaching statistics in high school concerns the crowded curriculum which exists there. The proponents of this argument maintain that there exists little room for consideration of a statistics course, even if it is an elective. This may be true if you disregard the relative importance that should be placed on the subject matter. However, the proper place for the teaching of statistics, or discussing them, is in the subject-matter field where they occur. If the curriculum is now satisfactory and statistical training is adequate therein, then there is little need for introducing a new course. There are few that believe this is a representative state of affairs and those who do cannot be naive enough to believe that very much correct and rigorous statistical teaching is going to be accomplished by having it taught by anyone who has not had considerable statistical training. Statisticians are inclined to believe that they should be the ones to teach statistics. This has much merit connected with it, and until the time (I fear in the far distant future!) when statisticians are graduated from colleges of education in comparable numbers to other majors or several statistics courses are taken by all college students, statisticians would rather surrender their just rights to mathematical teachers at the high school level than anyone else. The question then is where and how much should be taught in the mathematics courses. My tentative answer to this is: put some statistical training in all academic and especially mathematical training from first grade to twelfth.

Space in this paper does not permit the detailed elaboration of the topics which need to be considered. At least, the definitions and discussion should be made of the terms: "population," "sample," and "parameter." The two main problems of statistical inference—estimation and tests of hypotheses—should form a central part of all teaching. It is important for students to understand the function of the random process for selecting samples from populations and what effects methods of biased selections may have. Attention should be drawn to the difference between mathematical error (which refers to a mistake) and to statistical error (which refers to the vagaries of chance). The characteristics and utility of the commonly used distribution functions, i.e., the normal, binomial, and Poisson distributions, should be made clear.

The topics mentioned in the last paragraph cannot be presented with clarity by a teacher who has only a vague notion of the purposes and problems of modern statistical methods. New teachers of mathematics can be prepared for teaching statistics if their trainers recognize their needs and provide the appropriate requirements during their college years. To the other teachers, the words of Helen Walker in 1931 are as pertinent now as they were then.

> The situation is full of challenge for those teachers who like to leave the beaten path and adventure a bit, who are not afraid of the hard study necessary to prepare themselves for teaching in a new field, and who have a genuine interest in that type of social problem which can be approached by a quantitative study of mass phenomena. Such teachers will need first to make themselves thoroughly at home in statistical method, not merely with its elementary phases but with its spirit and some of its theory. It will be most unfortunate if teachers who have had only a six-weeks' summer course in statistical method are the ones who undertake this pioneering, because the selection of material for a simple course is not in itself a simple task, and cannot be well done by the person whose knowledge is elementary.[11]

3. An Example of a Simple Test of an Hypothesis

In order to show some of the character of modern statistics and provide an example of what a test of an hypothesis is like,

a complete discussion of a simple comparison hypothesis that uses ranked data will be given in this section. Tests of statistical hypotheses are the ultimate goal in properly conducted statistical training. Unfortunately this goal is seldom achieved by our secondary school students. The mathematics involved in this example will be found to be only arithmetic but some new vocabulary must be used. The problem will be set forth according to the outline of the scientific method given earlier and the *numbers* in the parentheses represent the steps of the scientific method.

(1) Suppose Johnny wants to find the quickest way to school from his home. There are two routes, A and B, which he wants to compare. (2) The form of the null hypothesis to be tested is that both routes take an equal amount of time. The term "null" is used in the sense of "no difference." This is the most simple hypothesis to test, both for logical and analytical reasons. (3) He next proposes to test the null hypothesis by keeping account of the time for each of six days when he goes to school.* In preparation for the execution of the experiment, Johnny realizes that there are many factors that influence the time en route. Some of them are the weather, the traffic, the time he leaves for school, how he feels, etc. It is almost impossible to eliminate any of these factors conveniently and the only way that their effects can be controlled is to *randomize* which route is taken on each day. This randomization is simply accomplished. He merely flips a coin for each day of the experiment. If the coin comes up "heads" he takes route A and if it comes up "tails" he takes route B. However, after three heads or tails are obtained, the routes for the other days are necessarily fixed.

* It is possible to determine, for a given accuracy, how many days should be considered in this experiment. The details are too difficult to be presented here. The number "six" has been chosen merely on the basis of convenience and simplicity.

(4) After six days the following data were collected:

Travel Time in Seconds and Their Ranks for Two School Routes

Route A

Day	*Time*	*Rank*
First	734	3
Fourth	841	6
Fifth	789	5
		14

Route B

Day	*Time*	*Rank*
Second	715	1
Third	773	4
Sixth	726	2
		7

Steps 6 and 7 in the scientific method will be completed after the analysis of the data which is Step 5. Statistical hypotheses are also tested in seven steps. We will now complete these for this particular problem.

A. First we state the mathematical form of the null hypothesis and what alternative hypotheses are going to be accepted if the null (the one under test) is rejected. The mathematical form of the null hypothesis for Johnny's problem is $A = B$, that is, the time of traveling is the same for each route. If this hypothesis is rejected then, of course, Johnny wants to accept the dictates of the data in saying whether $A > B$, or $A < B$, that is, whether A takes a longer or less time than B.

B. For the problems of the real world (with which statistics are concerned) almost all things are possible. The two routes may actually take the same (parametric) time, but due to the effects which we have attempted to randomize, the results of experiment may indicate that they are unequal and one route is better than another. The effects of these chance factors can be calculated under the assumption that the null hypothesis is true and results of these calculations are called *random sampling distributions*. All possible outcomes of an experiment are possible under the assumption that the null hypothesis is true and only random sampling errors are operating. If this is correct, how

are we ever to reject the null hypothesis? The statistician does this by saying that if the observed results occur by chance very infrequently then the observed results are not likely to be due to chance, but to the existence of some non-zero difference. To the word "infrequently" he attaches a definite probability, α, usually $.05 = 1/20$. Thus he says that if the observed results could occur by chance with probabilities less than or equal to .05, say, then real effects exist and the null hypothesis has to be rejected in favor of the alternative hypotheses. The object of Step B then is to specify the probability α. Of course the statistician realizes that in adopting such a course of action that true null hypotheses are being rejected in 5 out of every 100 of his analyses. For our problem, let us take $\alpha = .10$, for a reason which will presently appear.

C. Now we select some statistic to use to test the null hypothesis. For purposes of simple illustration we will use the ranks associated with the time of travel. Of course, modern statistical theory enables us to use the actual times, rather than their ranks, but the exposition of details of such analysis would necessitate more complication than I think is advisable now. The statistic, S_r, we will select is the sum of the ranks for the routes. Notice that we really need only consider one sum, say the smallest, since the other is fixed by the problem.

D. What is the random sampling distribution of S_r? In other words, what are the possible equally likely values of S_r and what is their probability of occurrence? Well, the minimum possible value of S_r would occur if one route had ranks 1, 2, 3 and, necessarily, the ranks for the other route would be 4, 5, 6. The sum of these two sets of numbers are 6 and 15, respectively. Therefore we have our range specified. We can construct all the possible, ${}_6C_3 = 20$, sums very simply for this little problem but for larger numbers of sums of ranks this becomes very tedious. This difficulty will be overcome by a table to be presented later. The possible 20 sums are:

1	1	1	1	1	1	1	1	1	1
2	2	2	2	3	3	3	4	4	5
3	4	5	6	4	5	6	5	6	6
6	7	8	9	8	9	10	10	11	12

2	2	2	2	2	2	3	3	3	4
3	3	3	4	4	5	4	4	5	5
4	5	6	5	6	6	5	6	6	6
9	10	11	11	12	13	12	13	14	15

These 20 sums may be collected in the array shown below, and the probabilities associated with each sum evaluated.

E. From the random sampling distribution we select a set of S_r called the critical region, for which the null hypothesis is to be rejected. The values chosen depend upon the alternative hypotheses. The S_r to be selected falls on the extremes of the distribution for both alternatives $A > B$ and $A < B$. In some problems only one alternative is of interest, then only one extreme is chosen. The probabilities associated with S_r of the critical region should equal or nearly equal α. For our problem, we entertain the possibility of both A being better than B and B being better than A. Looking at both extremes of the random sampling distribution,

Possible Sums:	6	7	8	9	10	11	12	13	14	15
Ways of Occurring:	1	1	2	3	3	3	3	2	1	1
Probability of Occurring:	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{1}{20}$
	.05	.05	.10	.15	.15	.15	.15	.10	.05	.05

$S_r = 6$ and 15, the probability associated with these two values equals $.05 + .05 = .10$ which is the size of the α. The reason we could not choose $\alpha = .05$ for this experiment is that we could not have both alternatives $A > B$ and $A < B$ and have the probabilities on the extremes add to 0.5. This is possible, however, if we increase the size of experiment to four trials for each route instead of three. To repeat, the critical values for Johnny's experiment are sums of ranks 6 and 15.

F. In this step we evaluate the criterion statistic for the experimental data. The minimum sum of ranks was found to be 7 which is the sum of ranks for route *B*.

G. At this point the researcher compares his experimental result with the critical region. As the observed value, 7, is not one of the critical values 6 or 15 the null hypothesis is accepted. Admittedly, the observed value is close to the critical value, but however tempting, it is not cricket to reject the hypothesis. We have set up the rules for behavior and must abide by them for the entirety of the steps of the scientific method. Therefore the action to be taken in the seventh step of the seventh step of the scientific method is for Johnny to use either route since we have no basis for rejecting the null hypothesis. What evidence there appears to be in favor of route *B* can be accounted for on the basis of random sampling error.

Had Johnny decided to use four trials for each method, the random sampling distribution would have looked as below.

The sum of the probabilities associated with the S_r of 10, 11, 25, and 26 is .056. In general, for these discrete random sampling distributions you can't always get exactly the α you want. In such cases the proper procedure is to select α by looking at the random sampling distribution *before* the sample statistic is compared with the critical region.

In 1945 Wilcoxon calculated the critical region values and probabilities nearest $\alpha = .05$ for various size experiments and we present part of his calculations at the top of page 559.

The type of problem just considered does not involve any complicated mathematics, but it does involve a new point of view for many pupils and teachers. The value of such a test of an hypothesis is self-evident and there is no better argu-

S_r:	10	11	12	13	14	15	16	17	18
Ways of Occurring:	1	1	2	3	5	5	7	7	8
Probability of Occurring:	$\frac{1}{70}$	$\frac{1}{70}$	$\frac{2}{70}$	$\frac{3}{70}$	$\frac{5}{70}$	$\frac{5}{70}$	$\frac{7}{70}$	$\frac{7}{70}$	$\frac{8}{70}$
	.014	.014	.029	.043	.071	.071	.100	.100	.114

(Continued)

S_r:	19	20	21	22	23	24	25	26
Ways of Occurring:	7	7	5	5	3	2	1	1
Probability of Occurring:	$\frac{7}{70}$	$\frac{7}{70}$	$\frac{5}{70}$	$\frac{5}{70}$	$\frac{3}{70}$	$\frac{2}{70}$	$\frac{1}{70}$	$\frac{1}{70}$
	.100	.100	.071	.071	.043	.029	.014	.014

Critical Values for Two Class Ranking Experiments with Two-Sided Alternative Hypotheses

*Number of Replicates**	*Lower and Upper S_r*	*Probability*
4	11, 25	.056
5	18, 37	.055
6	26, 52	.041
7	36, 69	.038
8	49, 87	.050
9	63, 108	.050
10	79, 131	.052

ment for such tests than a useful illustration. The principles employed there may be applied to many practical problems.

4. Conclusion and References

The purposes of this paper were to point out some of the nature of modern statistical methods and to encourage the teaching of these methods to secondary school students by mathematics teachers. These purposes were accomplished by a discussion of statistics, the case of teaching statistics in secondary schools, and a particular example of statistical method.

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* The number of replicates equals one-half of the highest rank in the analysis.

HAVE YOU READ?

Meserve, Bruce E., The University of Illinois. "List of Mathematical Competencies," *The School Review*, February 1953, p. 85.

In The Mathematics Teacher for February, 1952, you no doubt read Kenneth Henderson's report, "Minimum Mathematical Needs for Students in the College of Engineering." You will also want to read Mr. Meserve's description and viewpoint of the same study. Your understandings of the implications of the work in Illinois will mean more after studying this report. The program outlined here places responsibility for educating the future mathematician directly in the hands of the elementary and secondary mathematics teacher. However, it also gives these teachers complete freedom in developing a program which will produce the caliber mathematician needed in our society. Whether you agree or disagree you will find much food for thought in this article.—Philip Peak, Indiana University, Bloomington, Indiana.

Arithmetic for Majors?

By Jack D. Wilson

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Preliminary Statement

School administrators are now asking that prospective mathematics teachers be given a better preparation for teaching arithmetic and general mathematics. This paper outlines certain trends related to these requests and describes a course in arithmetic to give prospective teachers the kind of training now being demanded by school officials.

Significant Trends

In the twenties mathematics went on the defensive in the secondary schools. The traditional courses were attacked without mercy both by general educators and by laymen. Most Americans, it was said, will never use mathematics as taught in high school. "Algebra is nearly valueless for 90 per cent of all boys and 99 per cent of all girls" (11, p. 2). "It is taught under compulsion to millions who can never use it even in intellectual play" (11, p. 2).

Many educators saw only one solution—eliminate mathematics from the high school programs of all but the few. This was an alarming development, and it could not go unchallenged. Mathematics educators tried to reverse the trend by working hard to popularize general mathematics, a "definitely arranged and psychologically ordered course" utilizing the best materials from algebra, geometry, and trigonometry and unified by one or more of such central ideas as functionality and graphic representation. But it became evident in the thirties that general mathematics was not meeting with success.

The consequences of eliminating mathematics became apparent in the forties. Military men found inductees incredibly weak in basic arithmetic. College teachers found students unable to do the simplest arithmetical operations. Employers complained of employees with lost or unacquired skills. General educators themselves began to deplore the lack of training in basic mathematics. Said Bobbitt: (2, p. 276)

> It is the laymen who ultimately build and destroy civilization and, in doing so, carry the specialists down in the wreckage. . . . The continuance of civilization is contingent upon the enlightenment of the laymen. Let the mathematicians now do as good service in infusing laymen's education with accuracy as they have done with the special technologies and they will contribute an indispensable element that is yet lacking. . . . A heavy responsibility rests upon the mathematics people to make mathematics give its proper social service.

The "mathematics people," well aware of existing conditions, were already demanding reforms. Carnahan (7, p. 246) suggested a course of remedial instruction for all high school seniors found deficient in basic arithmetic. Buckingham (5, p. 212) asked for a compulsory high school course to relieve the crowded arithmetic program of the junior high school. Breslich (3, p. 293) demanded the measurement of arithmetical growth year by year over the entire school period.

With such general recognition of the problem and so many excellent suggestions for its solution, why has the problem remained unsolved?

A Possible Reason and a Suggested Solution

The success of any course—algebra, geometry, consumer mathematics, general mathematics, or arithmetic—is dependent on the attitude and effort of the teacher. The most carefully planned curriculum can be sabotaged in the classroom. Perhaps this explains, at least in part, the unpopularity of high school courses in arithmetic and in general mathematics.

Schorling (12, p. 152) found through his nationwide questionnaire that a very small percentage of teachers like to teach general mathematics. Gager (9, p. 298) stated that many administrators have permitted too many of their best teachers to scheme around or wiggle out of the responsibility of teaching general mathematics. Bartky (1, p. 16) observed that many mathematics teachers are failing in their duty by tossing out all weak students from their classes.

What underlying cause led to these findings and to these criticisms? Is it merely inertia, resistance to change? Is it the inherent difficulty of organizing suitable materials? Or may the answer lie in the specialized pre-service education of the mathematics teacher? Douglass thought so: (8, p. 214)

> Perhaps as influential as any other force is the questionable attitude taken by many teachers of mathematics that it would be beneath their dignity as majors in mathematics—students of calculus and differential equations—to teach mathematics other than algebra, geometry, and trigonometry in high school. I can well appreciate this position, its naturalness and its dangers.

Research (10, 12, and others) appears to support this conclusion.

Why does the good mathematics teacher "wiggle out of" the task of teaching general mathematics and arithmetic? Why does he dislike being assigned to teach these courses? One reason: he considers high school arithmetic only a drill course, a dull dry review of work already covered in earlier grades.

This need not be. There is a new and refreshing point of view in the field of high school arithmetic. Buswell contrasts this new point of view with the older concept: (6, p. 15)

> There has recently been some discussion of a quite different type of arithmetic at the high school level. A course in arithmetic in the ninth grade would have possibilities of furthering an understanding of the number system, which would be a logical sequence to the teaching which the elementary schools have given. As arithmetic is now taught in the elementary grades, most pupils do not gain a very clear understanding of the nature of the number system. If a high school course in arithmetic is to be effective, it must stake out new ground for itself. I suggest that this new ground might be found in a better understanding of the number system and a careful systematic study of arithmetical meanings. Such a course might contribute that understanding of arithmetic that is needed to make it function in quantitative thinking.

But young mathematics teachers will not be inclined to stake out, unaided, this new ground. The college itself has an obligation to outline the skeleton structure of the new arithmetic. Indeed, a special course seems to be indicated.

Our first attempts to organize such a course are described in the section which follows.

Course Description

Our purpose is not to teach a theory of numbers although some theory is included nor to engage in arithmetical computation although some computation is desirable. Rather, the objective is to develop a point of view so that the young instructor, assigned to teach a class in arithmetic, will accept the task, not as a chore or a penalty, but as a challenge and an opportunity. Selection of course materials is influenced by many considerations but the overriding theme, the unifying principle, can be stated in one sentence, "Number is a system; it should be taught as a system."

The birth of number is the first topic: man's limited number sense; primitive matching, such as ponies with stones, or knots with days; the first hesitant steps in counting up to five or ten; the development of number bases. Primitive number systems are analyzed—Egyptian, Babylonian, Greek, Roman, and Mayan—with emphasis on symbolism, formation of numbers, and extent to which place value and zero are utilized. And so to the Hindu-Arabic system.

Students of mathematics are so familiar with the decimal system and its many generalizations that they may be puzzled by and impatient with the gropings of the mathematically illiterate among their

pupils. What can be done to narrow this gap? What experiences will assist them to understand better the difficulties of the mathematically inept students who constitute such a large segment of the high school population? Buswell (6, p. 5) answers, "It is rather difficult to get a concept of the decimal system until one makes some comparison with number systems based on some number other than ten."

Accordingly, our students are required to work with bases other than ten, to build addition and multiplication tables and to compute using the new systems. To round out the picture, an opportunity is available to introduce the concepts of congruence and modulus and related notions and to require readings from books like Dantzig's *Number, the Language of Science* and Hardy's *Pure Mathematics*. Through such studies in number systems the student becomes aware of the nature of our own system and the difficulties experienced by children in mastering its major generalizations.

The course is not confined to these bounds. High school teachers should be familiar with the techniques now used to teach the decimal system in the elementary grades because these methods can also be employed effectively in senior courses. Accordingly, our students are required to study these modern techniques. They learn that several levels of instruction (object, picture, semi-symbolic, and symbolic) can be utilized to make learning more meaningful. They become familiar with the kinds, purposes, and uses of multiple aids—the abacus and place-value pockets to teach positional value and the function of zero, hundred boards to teach per cent, fractional parts to demonstrate operations with fractions—and various other visual aids designed to make arithmetic easy to understand. They make a careful study of those arithmetical meanings which are poorly understood by junior and senior high school students.

Further, all teachers of mathematics should have some knowledge of the revolution which has taken place in the teaching of arithmetic during the last thirty years. To illustrate, our students study and discuss the leading theories of instruction: (1) the drill theory, popular in the twenties, which insisted that the child be taught a great mass of comparatively separate and unconnected facts to be mastered by formal repetition in adult form whether or not understood; (2) the incidental learning theory, popular in the early thirties, which emphasized informal instruction given only when needed in real life activities; and (3) the meaning theory, popular today, which came to the aid of an arithmetic which seemed on the verge of destruction. Examination of these theories makes clear the drastic shift in thinking which took place between 1928 and 1938: (4, p. 330)

> Ten years ago all leaders in this field were atomists who pinned their faith to the doctrine of definite responses to definite stimuli. . . . They exalted specific skills which they sought to develop by drill according to the law of exercise.
>
> Today [1938], with almost equal universality, students of arithmetic adhere to some type of Gestalt psychology. They are . . . more convinced of the efficacy of insight than of the sufficiency of repetition, more concerned with understanding than with skill, and more than ever reverent of the deep mysteries of individual differences.

Through this reading and discussion students become oriented to "Arithmetic, 1953," its theory and practice. They are not so likely to commit the errors which made instruction in earlier decades unproductive and sterile.

Woven into the course are interesting topics selected from the historical, the recreational, and the computational aspects of arithmetic. Here are some illustrations. From history of arithmetic: obsolete methods of multiplication—lattice, scratch and peasant methods, and Napier's bones. From recreational arithmetic: games based on the binary number system and casting out nines. From computational arithmetic: Euclid's algorithm, including proof, and tests of divisibility. Interesting and unusual problems are introduced when

appropriate. An example is the following borrowed from the Stanford Competitive Examination:

> Among grandfather's papers a bill was found.
>
> 72 turkeys.....$—67.9—
>
> The first and the last digits that obviously represented the total price, replaced here by blanks, have faded and are illegible. What are the two faded digits and what the price of one turkey?

This problem can serve either as an introduction to or an application of the rules for divisibility.

Concluding Arguments

Some will object to inclusion of an arithmetic course in the program of the mathematics major or minor. Already there are too many other courses competing for a place in the curriculum. Still, is it really imperative that the teaching major receive a scholarly preparation in so many branches of advanced mathematics? Will one course more or less in tensor analysis or functions of a complex variable adversely affect the teaching competence of a secondary teacher? The writer thinks not.

Can a course in which the mathematics is admittedly so elementary be justified in the upper division? Using a single criterion, sheer difficulty, no! The student does not have to struggle as he would in solving a difficult differential equation. But difficulty is only one criterion. Few courses in education, for example, could qualify on this basis alone.

Why then is not the course included in the education program in the fifth college year? Is there not a curriculum and instruction course to develop the point of view and to describe and teach the materials suggested in this paper? The answer is "no"; a single course in methods can only touch on such content. If these ideas are worth developing, they must be taught in a separate course, preferably in the Department of Mathematics.

This brings us back to the fundamental question. Are the concepts really worth developing? There is no final answer, but a partial answer probably could be arrived at through systematic evaluation. How could this evaluation be carried out? Perhaps the only satisfactory method would be to poll former students who have had an opportunity to face reality in the classroom. So far this has not been done, but unsolicited comments have been coming in and they are uniformly favorable. "The most useful course in the program." "Worth two semesters of calculus." "Thank goodness for the arithmetic course." Despite this favorable reception, we recognize the possibility that some other organization of materials might better achieve our objective. It is hoped that those who have alternative suggestions will share with us the product of their thinking. Only in this way can we succeed in giving our students the kind of preparation needed for teaching all high school pupils, and especially the other 85 per cent.

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Leonardo da Vinci—A Pictorial Chart

A new pictorial chart shows the place of Leonardo da Vinci in the stream of history by illustrating the main events and works of his life in chronological relationship with other personalities and events of his time.

Horizontally, the chart is divided into three general sections. The center section has small reproductions of some of Leonardo's works, as well as pictures of the house where he was born and some of the buildings in which he lived or worked. Above this, important events of the period are noted, and below it are the names of some of Leonardo's outstanding contemporaries, as well as pictures of his four patrons and costumes of the day.

Leonardo's notebooks, written in minute and accurate left-handed mirror writing and filled with sketches of his inventions, ideas and improvements of already-existing devices, are crammed with such foresighted things as air conditioners, triple-tiered machine guns—one group of barrels could be fired while the second cooled and the third was being loaded—airplanes, anatomical drawings that can be used with profit today, designs for two-level highways for pedestrian and vehicular traffic, as well as that indispensable and perhaps too-familiar roadside instrument, the jack.

Among the ideas that came newly minted from Leonardo's mind, his aeronautical studies are the most intriguing, for they represent his obstacle-surmounting genius in its clearest form. He devised a parachute—or tent of linen, as he called it—that is said to have been used successfully. His airplane was the fruit of his scientific treatment of the problem of flight, and modern engineers say that it was theoretically sound. And we must bear in mind that he defined the now everyday principle of aerodynamic reciprocity 200 years before Newton.

We are all familiar today with the helicopter that reached easily into the mountainous areas of the Korean battleground to pluck the wounded out from under enemy fire. Well, Leonardo foresaw this aircraft, too. His aerial screw is the forerunner of our helicopter, and its design makes it the prototype of today's ship's propeller as well.

Among the things that held Leonardo back from the full realization of his inventive genius, which as we have seen far outstripped the facilities of his time, were, it has been said, a compact power unit and a metal sufficiently hard for his needs. Nevertheless, carried on in a spirit of free inquiry, his work has meant much to the generations that followed him.

Prepared by the Fine Arts Department of International Business Machines Corporation and printed in sepia, this study chart and a four-page mimeographed story of some of Leonardo's inventions and scientific studies are available free of charge. Write to the Department of Information, International Business Machines Corporation, 590 Madison Avenue, New York 22, New York.

The Engineer's Simple Arithmetic

By WALTER J. SEELEY

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THE WORK of the engineer in design, research, or development very often involves extensive arithmetical computations. For this he uses a slide rule, computing machine, and sometimes logarithms, but first the factors are set up in some convenient form for easy manipulation. In all his work the engineer is concerned with two things: short cuts to save time, and accuracy. Herewith is outlined a method which constitutes somewhat of a short cut and at the same time results in increased accuracy for arithmetical computation. It is a method much used by engineering students and practicing engineers; it is very convenient, and should be made available to high school students.

The method is really quite simple and can be taught to high school students who have had the theory of exponents in algebra. It also can be used to assist in the understanding of exponents. Examples will be used to illustrate the scheme, which merely is a judicious use of exponents.

Take any two numbers, say 46,800 and 0.00675, and write them thus:

$$46{,}800 = 4.68\times10^{4}$$

$$0.00675 = 6.75\times10^{-3}$$

This puts them in simple form for easy arithmetical manipulation in multiplication and division, and this is the first step in the process. All numbers are thus treated as simple multipliers into multiples of ten.

1. *Multiplication*

Multiplication is handled as follows:

$$46800\times0.00675$$
$$= (4.68\times10^{4})\times(6.75\times10^{-3})$$
$$= (4.68\times6.75)\times10^{(4-3)}.$$

First, an estimate of the answer is made by inspection. The product of the two numbers (not including the ten) will be somewhere around 30, which, multiplied by 10, will result in 300 as an approximate answer. The actual multiplication then can be carried out by any desired method, with a foreknowledge of the magnitude of the answer and the proper place for the decimal point.

2. *Division*

$$\frac{46800}{0.00675} = \frac{4.68\times10^{4}}{6.75\times10^{-3}} = \frac{46.8\times10^{3}}{6.75\times10^{-3}}$$
$$= \frac{46.8\times10^{(3+3)}}{6.75}.$$

Again an estimate by inspection will give about 7, to be multiplied by 10^{6} (or one million). Attention is called to the change of 4.68×10^{4} to 46.8×10^{3} in order to set up a convenient fraction for approximate solution by inspection.

3. *Division*

$$\frac{0.00675}{46800} = \frac{6.75\times10^{-3}}{4.68\times10^{4}} = \frac{6.75\times10^{(-3-4)}}{4.68}.$$

By inspection this will give about 1.5×10^{-7}, or 0.15×10^{-6}, or 0.15 divided by one million. 1.5×10^{-7} is a much more expressive and a much neater number than 0.00000015, and is preferred by engineers.

In all three of the above examples the exact numerical value, of course, must be obtained by conventional methods of multiplication or division. The approximate result is known, however, before the labor is performed, thus giving an idea of what to expect for an answer. Long-hand multiplication or logarithms may be used at will.

Logarithms

The performance of the above arithmetical operations by logarithms also is greatly facilitated by the same use of ex-

ponents. For illustration the same three examples above now will be solved by logarithms.

4. *Multiplication*

$$46800\times0.00675=4.68\times10^{4}\times6.75\times10^{-3}$$
$$=[4.68\times6.75]\times10.$$

Only the quantity in the brackets is solved by logs. The 10 outside stays there to be multiplied into the final answer. Thus

$$4.68\times6.75=\text{antilog of }[\log 4.68+\log 6.75]$$
$$=\text{antilog of }[0.6702+0.8293]$$
$$=\text{antilog of }1.4995=31.6.$$

Final result $=31.6\times10=316$.

Logs of only simple numbers are thus involved.

5. *Division*

$$\frac{46800}{0.00675}=\frac{46.8\times10^{3}}{6.75\times10^{-3}}=\frac{46.8\times10^{6}}{6.75}$$

$$\frac{46.8}{6.75}=\text{antilog of }[\log 46.8-\log 6.75]$$
$$=\text{antilog of }[1.6702-0.8293]$$
$$=\text{antilog of }0.8409=6.93.$$

Final result $=6.93\times10^{6}$
$=6.93\times1{,}000{,}000.$

Notice how 46,800 was manipulated in order to avoid the negative characteristic. This is important.

6. *Division*

$$\frac{0.00675}{46800}=\frac{6.75\times10^{-3}}{4.68\times10^{4}}=\left[\frac{6.75}{4.68}\right]\times10^{-7}$$

$$\frac{6.75}{4.68}=\text{antilog of }[\log 6.75-\log 4.68]$$
$$=\text{antilog of }[0.8293-0.6702]$$
$$=\text{antilog of }0.1591=1.44$$

Final result $=1.44\times10^{-7}$.

By this method negative characteristics, with their extreme puzzlement to the student, are entirely eliminated. This has merit. The student no longer has to worry about the correct application of $9\ldots-10$. Logarithmic computation thus may be taught without ever having to mention that there is such a thing as a negative characteristic. And the improvement in accuracy is obvious.

One final example of something a little more complicated will be given.

7. $$\sqrt[3]{\frac{46800}{0.00675}}=\left[\frac{46800}{0.00675}\right]^{1/3}$$
$$=\left[\frac{46.8\times10^{3}}{6.75\times10^{-3}}\right]^{1/3}$$
$$=\left[\frac{46.8}{6.75}\right]^{1/3}\times[10^{6}]^{1/3}$$
$$=\left[\frac{46.8}{6.75}\right]^{1/3}\times10^{2}$$

$$\left[\frac{46.8}{6.75}\right]^{1/3}$$
$$=\text{antilog of }\tfrac{1}{3}[\log 46.8-\log 6.75]$$
$$=\text{antilog of }\tfrac{1}{3}\times0.8409$$
$$=\text{antilog of }0.2803=1.91.$$

Final result $=1.91\times100=191$.

These simplifications are in common everyday use by engineers. They result in easy manipulation and good accuracy. They could be taught to high school students and should result in enhanced interest both in computation and in the use of exponents.

Note: An accurate and simple log table not requiring interpolation is *A Graphic Table of Logarithms* by Lacroix and Ragot, published by The Macmillan Co., and the price is reasonable.

"I think mathematics has the advantage of teaching you the habit of thinking without passion. That seems to me the great merit of mathematics. You learn to use your mind primarily upon material where passion doesn't come in, and having trained it in that way you can then use it passionlessly upon matters about which you feel passionately. Then you're much more likely to come to true conclusions."

—Bertrand Russell, *Spinoza's Ethics* (1942).

APPLICATIONS

ERNEST R. RANUCCI, *Guest Editor*
Weequahic High School, Newark, New Jersey

IN THE PAST I have found that the seventeenth yearbook of the National Council of Teachers of Mathematics, *A Source Book of Mathematical Applications*, has been a wonderfully helpful volume in keeping mathematics students interested. I am adding several examples of the geometry connected with carpentry to the many found in the seventeenth yearbook.

Ar. 33 Gr. 7–9. The Phillips recessed head screw is being used more and more in industry. Instead of the normal slotted screw, this one has two slots at right angles to each other. It is usually used with a special screw driver whose blade matches the pattern of the right-angled slots. The special slotting results in obtaining a greater bearing surface between screw driver and screw with resulting ease of operation. Time is also saved since the screw driver never has to be turned more than 45° to match slots.

P.G. 11 Gr. 7–11. Ceiling tile has become a favorite with carpenters, both professional and week-end, since the material is cheap and easily installed. It is usually applied right over an old ceiling and comes in two basic forms, in single squares and in two squares joined together. In either case the edges of the tiles are constructed in such a way that a protruding lip on one of the tiles overlaps the edge of the adjacent tile, thereby concealing nail-heads. Single tiles may be applied in a normal grid pattern. The difficulty here is that when four adjacent tiles come together at a corner a poor fit will result if any of the tiles has been driven in too tightly. The sum of the four right angles involved sometimes does not quite make the desired 360°. Poor fits like these have a tendency to pile up with the result that some line of tiles at a later stage of the work may be out as much as an inch or two. The week-end carpenter avoids this difficulty by offsetting each row of tiles by just one-half of a tile's width. Thus the geometric aspect of the problem no longer involves four right angles but three angles, two of 90° and one of 180°. Butting two right angles against a straight angle causes no spatial problems. A similar type of offsetting may be used with the double tile but here the finished product will show the normal grid pattern.

Ar. 34 Gr. 7–9. Double doors such as are found on television and radio cabinets require a great deal of fitting before hinges, latches, etc. operate properly. Much of this labor may be saved by cutting and applying the double door in one piece. Hinges are applied in the usual manner to both ends of the one-piece door. After this, the door may be cut into two parts. The process guarantees that the separate doors will butt properly, with a minimum of fitting. The geometric process of cutting up a rectangle into two smaller ones is simpler than the reverse process of building up two rectangles to equal a third. Another advantage of the method is that a continuous grain between two doors is obtained, a feature which would require a high degree of craftsmanship if two doors were worked on and each finished separately.

The President's Page

Edited by JOHN R. MAYOR
The University of Wisconsin, Madison 6, Wisconsin

TWO NEW PUBLICATIONS

IN THE SPRING of 1954, The National Council of Teachers of Mathematics will publish two new important periodicals, for which we believe there has been a long-time need, and which have been under consideration by the Council for a number of years. These new publications, and their editors, are:

1. *The Arithmetic Teacher*
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   Associate Editor:
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   University, Alabama

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   H. L. MEYER, JR.
   The University of Chicago
   Chicago, Illinois

Announcement of these periodicals is also to be found in the advertisements on the opposite page to which your attention is called.

Members of the National Council will be well aware that these publications will serve the worthy purposes for which they are intended only if they reach readers who can profit most from them. This new venture imposes a new and special responsibility on each member of the National Council to make certain that teachers and students in his school know about these publications and that interested persons subscribe for them. A secondary reason for accepting this new responsibility comes from the financial side of the venture, since the National Council with with its present low membership fee cannot provide these new services unless they very soon become self-supporting. The self-imposed responsibility for assistance of readers of this page in "selling" these new services to those who can best profit from them will be the greatest factor in the success or failure of these new Council activities. Your help is essential and we are confident that we can depend upon you.

The Arithmetic Teacher, a quarterly journal, is to be devoted to the improvement of the teaching of mathematics in kindergarten and the grades of the elementary school. While the periodical will be of most direct value to teachers at this level, this journal should also be of considerable interest to teachers in more advanced grades. Secondary school teachers can always profit from better understanding of the problems of the teaching of arithmetic. While THE MATHEMATICS TEACHER will continue to carry a limited number of articles of interest to teachers in the primary and intermediate grades, certainly a much more adequate coverage of important topics in this area will be made possible by the new journal. As one interested in the teaching of mathematics at all levels of instruction, I am happy to give a strong recommendation of this new periodical to all mathematics teachers.

The primary purpose of *The Mathemat-*

(Continued on page 570)

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Pres.—Mrs. W. K. Holmes, Dunbar
Secy.—O. C. Hess, Harrisville
Treas.—Ruth Dunn, Richmond

Wisconsin Mathematics Council
Pres.—Sister Mary Felice, Mount Mary College, Milwaukee 10
V.-Pres.—Lydia Goerz, Mary D. Bradford High School, Lempsha
V.-Pres.—Dorothy Sward, Junior High School, Beloit
V.-Pres.—Mrs. Mary Samter, 109 N. Division, Stevens Point
Secy.—Ruth Conlin, 443 West Dayton, Madison
Treas.—A. A. Vorba, Marshfield High School, Marshfield

Wyoming Association of Mathematics Teachers
Pres.-Secy.-Treas.—Justin O. King, Rawlins

FUN

Why is simplifying a fraction like powdering the nose? It improves the appearance without changing the value.

Daffynition: An exponent is a small number written in the northwest corner of another number.

Arithmetic teachers sure know how to make the little things count.

Teacher: "Tommy, if you had six apples and I asked for three, how many would you have left?"

Tommy: "Six."

HISTORICALLY SPEAKING, - -

Edited by PHILLIP S. JONES
University of Michigan, Ann Arbor, Michigan

The Binary System

The binary system as a special case of the generalized problem of scales of notation has had a sudden resurgence of popularity. This is largely due to its use in modern high-speed electronic calculators and in new developments in the theory of "information" and "communication."[1] However, this new utility of the binary system arrived at the same time that an even greater emphasis was being placed on "meaning" and "understanding" in the teaching of mathematics. In arithmetic (and algebra) many teachers have felt that understanding of our number system was enhanced, and in some cases first achieved, through a study of numbers written to some base other than ten. These two motives, utilitarian and pedagogical, have led to several articles on the history of the binary system and related topics,[2] but it seems that none of them have stressed several additional pedagogical values to be derived from a proper survey of the historical background of scales of notation. This topic is not only intrinsically interesting, but it also illustrates well the role of generalization and abstraction in mathematics, the roles of necessity and intellectual curiosity in mathematical invention, a few of the many connections between mathematics and philosophy and religion, and the interesting phenomenon of simultaneity in discovery which recurs so often in the history of mathematics.

The binary system is associated at once with both the most primitive, perhaps prehistoric, mathematics and some of its newest developments. Prehistoric mathematics may be studied either through the efforts of the archaeologists or through those of the anthropologist who studies primitive contemporary societies and from them extrapolates backward, not exact historical data, but notions of how ideas developed. Archaeologists have little to tell of prehistoric number systems although their existence seems quite certain from the fact that the early written records include substantial systems of numeration. Anthropologists and other scholars, on the other hand, have gathered much data on primitive number systems. Although the most common base in such systems is *ten*, many others have been used in various parts of the world. *Five*, *twelve*, *twenty*, and *sixty* have all been used frequently, but the systems based on *two* are probably the most rudimentary found. For example, only two number words were found to exist among the western tribes of the Torres Straits. Their numbers were 1—*urapun*, 2—*okosa*, 3—*okosa urapun*, 4—*okosa okosa*, 5—*okosa okosa urapun*, 6—*okosa okosa okosa*, more than 6—*ras*.[3]

[1] Gilbert W. King, "Information," *The Scientific American*, Vol. 187 (September 1952), p. 132 ff.

[2] Phillip S. Jones, "Miscellanea—Mathematical, Historical, Pedagogical," THE MATHEMATICS TEACHER, XLII (1949), 306.

Vera Sanford, "Notes on the History of Mathematics," THE MATHEMATICS TEACHER, XLIV (1951), 29, 30.

John E. Milholland, "Card Sorting and the Binary System," THE MATHEMATICS TEACHER, XLIV (1951), 312–14.

Martin Hirsch, "An Example of the Method of Duplation," THE MATHEMATICS TEACHER, XLIV (1951), 591.

Several of these articles contain excellent additional references to which we add Louis N. Ridenour, "The Role of the Computer," *The Scientific American*, Vol. 187 (September 1952), p. 116 ff.

[3] Levi L. Conant, *The Number Concept* (New York: The Macmillan Company, 1931), p. 105.

Of course this number "system" does not use the place-value notion nor is it even entirely consistent in using the base 2. To have been consistent a new name for two groups of two each would have been needed in writing 4, 5, 6 just as we use *hundred* for ten groups of ten each. Although it is not quite fair to call this a binary system nor to accept anthropological evidence of this sort without reservation as indicating what happened in prehistoric days, still such ideas have an intrinsic interest as well as indicate how notions of number must have grown slowly and incompletely out of the communication, trade, and record-keeping needs of early men.

The essential notions of what most properly should be called a "system of numeration" rather than "number system" include in addition to a *base* and *place value*, a *zero*, *symbols for the digits*, and a (decimal) "*point.*" It is not our task here to discuss the development of these concepts although in teaching they might well be dwelt upon, especially the sexagesimal system of the Babylonians from which we derive our units of time, angle, arc, and also the special role of zero, the logical and historical development of decimal fractions and their relations with fractions in general, computation, and the concept of irrational number. However, man cannot be said to have really comprehended either number or a system of numeration until he has realized the arbitrariness of the base and has experimented with writing and computing with various other bases, i.e., until he has generalized the notion of a base and formulated it abstractly. This abstraction waited many years for persons prodded by intellectual curiosity, and then again, years later, these generalizations were found to be the tools needed in such modern developments as we cited in our opening paragraphs.

Lucas attributes, rather dubiously, to Aristotle (350 B.C.) the notion that *four* could equally well replace *ten* in numeration,[4] and further states that Simon Stevin, the originator of our decimal fractions (*La Disme*, 1585) also proposed a duodecimal system. However, these were partial approaches. The earliest evidence of a real and complete grasp of the arbitrariness and generality with which a base may be selected has only recently been discovered in the manuscripts of Thomas Harriot, the English algebraist who, sent in 1585 by Sir Walter Raleigh to survey Virginia, was probably the first mathematician of any importance to touch the Western Hemisphere.[5] (It is somewhat interesting to note that Lucas' chapter on "La Numeration Binaire" is dedicated "à Monsieur J. J. Sylvester, correspondant de l'Institut, professeur de l'Université J. Hopkins à Baltimore," a great English mathematician who lived and taught here for some time.)

Previously credit for a generalized treatment of binary arithmetic has usually been assigned to G. W. Leibniz, German philosopher and developer of the calculus. He still retains the distinction of having written the first *publication* on the *binary* system in 1703.[6]

However, Blaise Pascal in an article, "De Numeris Multiplicibus," supposedly written in 1654 but published posthumously in 1665 discussed various tests for divisibility, including what we would call the check by nines.[7] In this he makes use

[4] Édouard Lucas, *Récréations Mathématiques* (Paris: 1891), p. 146.

[5] John W. Shirley, "Binary Numeration before Leibniz," *American Journal of Physics*, Vol. 19 (1951), pp. 452–54.

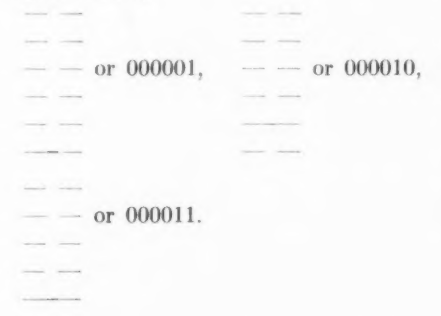
[6] *Leibnizens Mathematischen Schriften* (ed. by C. I. Gerhardt) (Halle: 1863), Vol. 7, p. 223 ff. reprints "Explication de L'Arithmetique Binaire—" from the *Memoires de l'Academie des Sciences*, 1703, and three manuscripts from the Royal Library of Hanover where he worked. Louis Couturat, in *Opuscules et Fragments Inédit de Leibniz* (Paris: 1903), pp. 278, 574, cites a letter on this subject written March 15, 1679, as well as a later letter and an essay.

[7] Blaise Pascal, *Traite du Triangle Arithmétique avec quelques autres petits traitez sur la mesme matière* (Paris: 1665), pp. 42–48. The date 1654 was given by Johannes Tropfke, *Geschichte der Elementar-Mathematik* (Berlin: 3rd edn., 1930), p. 134.

of the idea of writing numbers to different bases and suggests that in a duodecimal system there would be a check by *elevens* corresponding to the check by nines in a decimal system. In this connection he notes that the decimal system is an "institution of man and not a necessity of nature," and that his procedures would work in any system.

Contemporary with Leibniz and Pascal, Johann Caramuel y Lobkowitz published in his *Mathesis biceps vetus et nova* (Campaniae: 1670) discussions of systems based on 3, 10, 12, 60.[8] It has been speculated that Leibniz himself may have had some of his ideas from his teacher Weigel who published *Arithmétique Tetractique* in 1689.[9] Here again, although there may have been some cross fertilization, we note the phenomenon of simultaneity.

Leibniz advanced the binary system as an interpretation of semi-mystical symbols found in an ancient Chinese work, *I-King*, attributed to a legislator Fohi. In this appeared a set of 64 symbols each composed of six lines, some of which were broken in the middle and some of which were not. Leibniz associated the six lines with the first six places in a binary system of numeration. A broken line he interpreted as *zero* and an unbroken line as *one*. Thus, 1, 2, 3 would have been represented respectively by



Philosopher Leibniz also saw in the fact that *all* integers could be represented binarily using only the two symbols, 0 and 1, a mathematical parallel to the fact that God (1) created the universe out of nothing (0). He also saw the binary system as parallel to several concepts in logic and philosophy.

As noted earlier, all of these abstractions and generalizations were motivated by intellectual curiosity rather than by practical applications or needs, yet unanticipated uses are now being made of the theory.

Somewhat intermediate between intellectual curiosity and modern practical application are the relationships of the binary system to Egyptian and "Russian peasant" systems of multiplication,[10] to weighing and card-guessing puzzles (Lucas[11] calls the latter "L'Eventail Mysterieux"), the Tower of Hanoi, and the game of Nim,[12] and to perfect numbers.[13]

All these relationships add interest and insight as well as fuel for "projects" and "reports." Still interesting but less important and well known are the connections between the binary system and Braille writing for the blind and a system of stenography published in 1898 by G. Peano, a mathematician famous for his work in mathematical logic and symbolism.

In closing this account of the binary system as an example of generalization and abstraction and of motivation in mathematics, we mention one number scale which is frequently discussed today, the duodecimal. The names of Stevin, Pascal, Caramuel have already been associated with it. G. L. Buffon expounded further on it in his *Essai d'Arithmétique Morale* (Paris: 1777). Perhaps someone will send us a more complete account of its history and the reasons for its study and recommended adoption.

[8] Tropfke, *op. cit.*, p. 4.

[9] Lucas, *op. cit.*, p. 146; Couturat, *op. cit.*, p. 278.

[10] Sanford, *loc. cit.*; Milholland, *loc. cit.*

[11] Sanford, *loc. cit.*; Lucas, *loc. cit.*

[12] Milholland, *op. cit.*; Lucas, *loc. cit.*

[13] Walther Lietzmann, *Lustiges und Merkwürdiges von Zahlen und Figurin.* (Göttingen: 7th ed., 1950), pp. 170–185.

Accounts of these matters may also be found in current English publications on recreations and number theory.

DEVICES FOR A MATHEMATICS LABORATORY

Edited by EMIL J. BERGER
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Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing of the aid for publication. Or if that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

A SIMPLE DEVICE FOR ILLUSTRATING THE SIZE OF A CUBIC YARD

The teacher who would like to have a model of a cubic yard at his disposal to assist pupils with the problem of visualizing the actual size of a cubic yard may be pleased to learn that the problem of storage is not insurmountable. The device suggested here is inexpensive, easily made, and readily collapsible for storage.

To construct this device procure twelve wooden yardsticks of the kind ordinarily used by hardware stores and lumberyards for purposes of advertising. The approximate dimensions of such yardsticks are usually $1\frac{1}{4}'' \times 36' \times 3/16''$. Other materials needed include eight pieces of angle iron $1\frac{1}{2}'' \times 1\frac{1}{2}''$, sixteen $3/16''$ stove bolts each $\frac{3}{4}''$ long, and a wing nut, flat washer, and lock washer for each of the sixteen bolts.

In order that the directions for assembling the cubic yard will not turn out to be needlessly confusing, suppose we identify four of the yardsticks with the letter A, a second set of four with the letter B, and the remaining set of four with the letter C.

Drill $3/16''$ holes near both ends of the four yardsticks marked A; the centers of these holes must be $1\frac{1}{4}''$ from the ends of the sticks. Then slip flat washers over the ends of four bolts and fit them into the holes just drilled. Note that the washers must be between the wood and the bolt heads.

Next drill $3/16''$ holes $5/8''$ from both ends of the eight yardsticks marked B and C. Insert washers and bolts in the ends of the set marked B in the same way as was indicated above for set A.

Now select one pair of sticks from set B and a second pair from set C. Join them together to form a square by fitting the bolts already in the ends of the two sticks marked B through the holes in the pair marked C. Finally place an angle iron and lock washer over each of the four protruding bolt ends and then make the square rigid by turning a wing nut down on each bolt. (See Fig. 1). Note that for sticks B and C the bolts are passed through the "inside" holes in the pieces of angle iron. A second square may be completed in the

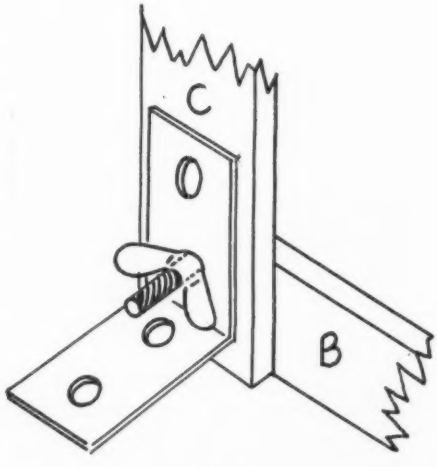

FIG. 1

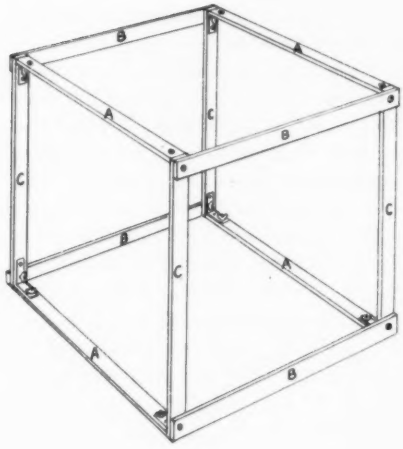

FIG. 2

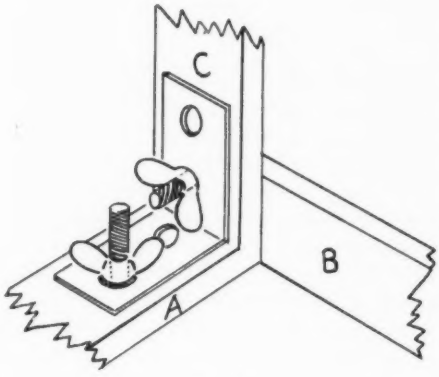

FIG. 3

same way by making use of the four remaining sticks from sets *B* and *C*.

Before continuing with the directions for finishing the construction notice that each of the two squares may be collapsed into elongated packages for storage merely by pushing together a pair of nonadjacent vertices.

Once the squares are formed, the cube may be completed simply by placing the set of yardsticks marked *A* in the positions shown in Figure 2 and fastening them as indicated in Figure 3. Note that the bolts are fitted into the "outside" holes of the angle irons this time. Figure 2 shows the completed device. To store the model it is only necessary to remove the yardsticks marked *A* and collapse the squares as suggested above.

SIGFRID E. ANDERSON
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A LEARNING AIDS SET MADE FROM TIN STRIPS

The following device might possibly be described as a homemade building set. It was designed to assist junior high school students in improving their understanding of some of the geometrical concepts normally studied at this level.

The set consists essentially of strips of charcoal tin plate (1-c) or tin-coated sheet iron about 26-gauge thickness. (See Fig. 4.) Strips may be cut to any desired length; the particular lengths shown in the illustration are simply the ones which this contributor has found convenient. All strips are finished in the same way. Simply cut the metal into strips 1″ wide with a tinner's shears and fold over both long edges into double hems with a bar folder. The shop teacher in your school will undoubtedly be able to advise you how to proceed with this operation. After the folding is completed, sand the rough-cut ends so that students who handle the pieces will not cut themselves.

Holes may be punched in the strips at various intervals as desired. The distances between holes indicated in the drawings of Figure 4 were decided upon somewhat ar-

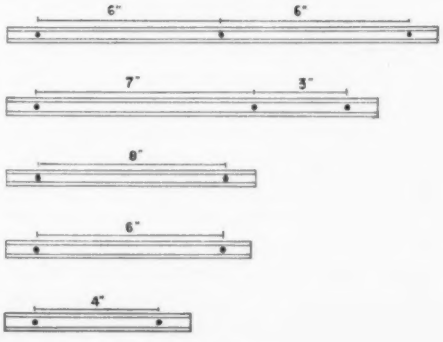

FIG. 4

bitrarily, although the idea was to give as much variety of arrangement as possible. The 1″ space between ends and holes prevents what has been called "bob-tailed geometry." Strips may be assembled to form angles, triangles, quadrilaterals, etc. by fastening them together with small machine bolts 1/8″ in diameter and $\frac{1}{2}$″ long.

Two or more pieces can be assembled to illustrate perpendicular lines, parallel lines cut by a transversal so as to form pairs of corresponding angles, vertical angles, complementary angles, and supplementary angles. Pieces can also be put together to form different kinds of triangles, squares, rectangles, parallelograms, etc., and to show that the triangle is a rigid figure.

A favorite laboratory exercise of this contributor is to ask students to form angles with strips, to estimate the sizes of the angles thus formed, and then to verify their estimates with the aid of a protractor.

R. Margaret Herr
Reynolds Junior High School
Lancaster, Pennsylvania

A Circle Device for Demonstrating Facts Which Relate to Inscribed Angles

Occasionally when students are invited to develop their own learning aids they come up with constructions which while not completely acceptable in the Euclidean sense sometimes work out better mechanically than those that are. That happens to be the case with this particular device.

A sophomore boy was asked to build a device which could be used to illustrate the fact that parallel lines cut off equal arcs on a circle. In terms of Figure 5 the obvious construction would, of course, have been to fix the position of some line such as AB parallel to the line of centers OO' of the two circles and then to develop a collapsible parallelogram $CDOO'$ by introducing movable parts for CD, OC, and $O'D$. The sophomore student understood all this, but he was determined to improve on the method just described for keeping AB parallel to CD.

For his particular purpose he selected a piece of fiber board 14″×26″×$\frac{1}{4}$″ on which to mount the movable parts he planned to use. Then he drew two 11$\frac{1}{2}$″ circles on the mounting board and divided them into quadrants by drawing a diameter in each perpendicular to their line of centers extended. (Note the three dashed lines in Fig. 5.)

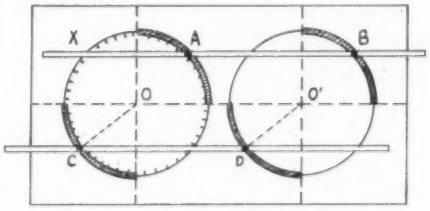

Fig. 5

With the aid of a drill and jigsaw he cut out two slots 3/16″ wide along opposite quadrants in each circle. Figure 5 indicates which quadrants are cut through. Note that the circular slots are cut along quadrants that occupy corresponding positions with respect to the two circles. This done, he drilled holes in two wooden bars AB and CD at points A, B, C, and D in such a way that $AB=CD=OO'$. Finally he fastened the bars to the mounting board by passing $\frac{1}{8}$″ machine bolts through the holes in the bars and the quadrant shaped slots and securing them loosely on the back side with washers and wing nuts. In order to be able to illustrate the truth of the theorem originally stated he divided the circumference of circle O into 72 equal arcs.

The amazing thing about this device is that the bars AB and CD always remain parallel to the line of centers and to each other when either or both are moved in such a way that the bolts slide along the grooved quadrants, and this movement is easy regardless of where the pressure is applied on a bar. In fact the property of parallelism required for bars AB and CD is effected much more nicely with this con-

struction than with the parallelogram construction suggested above in the second paragraph. In addition the grooved construction developed by the sophomore student makes it possible to involve both bars in the illustration desired.

The question about the new construction of course is: What argument can be developed to show that AB and CD are always parallel for all positions that they can be made to assume? We can think of none better than the one stated by the boy who produced the device.

"Suppose," he said, "that we move bar CD into a starting position so that points C, D, O, and O' lie in a straight line. Then if we move bar CD downward to the right both C and D will travel equal distances along their respective grooves and bar CD will remain parallel to OO'. Similarly with AB." This argument is of course valid only if points C and D always move over equal distances along their respective grooves at the same time. Mechanically this actually appears to be the case. So suppose we assume this fact. Then the answer to the question stated above should be obvious.

If points A and C are joined by a piece of black elastic, then the device may be used as the figure for proving the theorem which it was designed to illustrate intuitively.

If a protractor is mounted along CD so that its vertex is at C, then the device may be used to illustrate the fact that an inscribed angle is equal to one-half of its intercepted arc. Move bar CD into the position of tangency for the two circles and the device may be used to illustrate that an angle formed by a tangent and a chord is also equal to one-half of its intercepted arc.

Turn the wing nut down on the bolt at A so as to fix the position of bar AB, add a second piece of elastic between C and X, and the device may be used to illustrate that all angles whose vertices lie on XCA and whose sides pass through points X and A are equal.

The last few suggestions have been included in this article to show what invariably happens when devices are planned to illustrate some particular fact. Slight modifications, adaptations, or additions to the original construction often make the device suitable for illustrating additional related facts.

One final note seems worth mentioning. The particular design of the construction described in this article is of interest, entirely aside from the fact that it is somewhat clever mechanically. The point of interest is that it was a beginning student of mathematics—one not yet inhibited by previously developed arguments—who thought it through. It is doubtful that a more advanced student would have allowed himself so much independence from previously acquired conclusions. Probably nothing illustrates better than classroom experience with the laboratory method how surprisingly little opportunity we actually do provide for students to learn to think independently.

E. J. B.

HAVE YOU READ?

"Mathematics in School and College," *The American Mathematical Monthly*, June–July 1953, page 380.

This editor's extract from the book *General Education in School and College* presents a challenging program for the critical selection and profitable extension in the content of high school mathematics. Loving elaboration of familiar ideas is a waste of time. This time should be spent in the development of the basic ideas which lie outside of and beyond the mechanical manipulations. These things are taught better and learned better in high schools than in colleges.—PHILIP PEAK, Indiana University, Bloomington, Indiana.

WHAT IS GOING ON IN YOUR SCHOOL?

Edited by

JOHN A. BROWN
Wisconsin High School
Madison, Wisconsin

and

H. T. KARNES
Louisiana State University
Baton Rouge, Louisiana

OUR BANKING AND MONEY SYSTEM

Synopsis

OUR "Banking and Money System" unit began with an overview of the whole topic. This was followed by discussions, problems, and explanations of large units of the banking system, such as checking and savings accounts, borrowing money, safety in banking, and banking personnel. Much time was spent on vocabulary drill because of questions that were aroused in the students' minds as they read, worked, and discussed their problems. The text used was *Making Sure of Arithmetic*, Book 8, by Morton, Gray, Springstun, and Schaaf. We used as references: *Arithmetic We Use*, Book 8, by Brueckner, Grossnickle, and Merton, and their new text, *Knowing with Numbers*.

A trip to one of the local banks and a lecture by the cashier cleared up many of the otherwise hazy concepts drawn from reading. We spent about one and one-half periods at the bank, getting firsthand the work and experiences of bank employees.

This radio script is the culminating activity of the unit.

Banking and Our Money System

ANNOUNCER (*Introduction*): The eight-to-three class of Lincoln School of Shawano, Wisconsin, is here today to present a discussion on "Banking and Our Money System." This unit of mathematics work has been under the direction of Miss Peg McCabe, who is our moderator.

MODERATOR: My mathematics class has recently completed the unit of which Larry spoke, so we have come to hold our culminating unit on the air so that our many listeners may hear what we have studied and learned.

Our money system began long ago and has developed from the barter stage to our present convenient system. Will you give a short summary of this progress, Gordon?

GORDON: In early times people exchanged grain, cattle, salt, and spices for things they wanted. In early bartering stages there was no standard unit of measure for value, so each transaction was argued over until both parties agreed. In colonial Virginia tobacco was used as a standard and became money to them. Beaver skins were used in the Northwest. In many coastal places fish and shells are used as money. Our present system of coins and paper money has grown from the need for a convenient medium of exchange.

MODERATOR: Karen, can you tell us what types of money we have adopted?

KAREN: We have coins made of several kinds of metal to represent amounts less than one dollar, and paper money, backed by gold and silver, in $1, $5, $10, $50, and $100 denominations. These coins and bills are called currency.

LORNA: Miss McCabe.

MODERATOR: Yes, Lorna.

LORNA: May I add that we also have credit instruments, such as checks and bank drafts, that serve as money and are used in a large percentage of our

financial exchanges? Promissary notes, too, are a credit instrument, but we'll talk more about these later.

PAT: Let's talk about our 15,000 American banks and the 410,000 skilled workers employed in them.

MODERATOR: Jill, how important is banking in our community?

JILL: Banking is one of our most important public services. It is used by nearly everyone every day. One of our authors said, "Banks are the *heart* of our great money system, and through *them* is pumped the billions of dollars which constitute the blood stream of American business." The many services our banks give us keep professions, industry, and business going at their required high level.

MODERATOR: What types of banks did we study, Carol?

CAROL: We studied state banks, which are given the right to do business by the government of the state in which they are located. Of our 15,000 banks in America, 10,000 of them are state banks and the other 5,000 are national banks. The national banks have been granted a federal charter or are authorized by the national government to do business. Of these banks, 6,700 belong to the Federal Reserve System. This is a system of bankers' banks, and each of the 12 Federal Reserve Banks offers services to the banks in its geographical area. Ours is located in Chicago.

MODERATOR: Now that we have talked about the kinds of banks, let's list the services our banks offer us. Do you have a list, Barbara?

BARBARA: Yes, I have what I think is quite a complete list.

1. A bank keeps money safely for all depositors.

2. It provides a means of making payments by check.

3. It puts the deposited money to work to earn money by lending it to those who need it.

4. It offers advice to newcomers as to the business possibilities in the area.

5. By the depository, it allows businessmen to drop their money into the bank after banking hours for safekeeping overnight.

6. It insures all deposits up to $10,000 against theft, embezzlement, etc., so the depositor can rest assured his money is safe.

7. It urges all to use its savings account plan and the safe deposit boxes.

MODERATOR: How can you become a depositor, Jon?

JON: By taking your money to the cashier or teller and asking to have it placed in either a savings or checking account. They will ask you to sign a card with the signature you will use on all legal papers.

MODERATOR: Joanne, which of the services most closely affect you people at your present age?

JOANNE: I believe it would be the savings account. Many of us have been given gifts of money or have deposited money ourselves in such accounts. These earn interest and so grow automatically without the depositor's being aware of it. The amount may be withdrawn when the depositor asks for it. Children, in order to withdraw theirs, must be accompanied by their parents.

MODERATOR: Now let's talk about the process of borrowing money. Everett, do you want to begin?

EVERETT: Yes, I'll begin by saying that a promissary note is a legal promise signed by the borrower by which he agrees and promises to repay the amount borrowed, plus the interest as stated on the note. There are two main kinds of promissary notes. They are time notes and demand notes. If you sign a time note, you are given a specified time when the money must be paid back. A demand note has no time limit but must be paid when the lender demands it. There are several special kinds of time notes, but the best type for your loan will be chosen by your banker. For instance, an install-

ment note is one paid by several payments rather than all at once.

MODERATOR: David, what is the name given to the amount of money borrowed?

DAVID: That is the principal. While you keep this money you must pay a sort of rent on it, called interest. The rate or per cent of the principal that is to be paid to the lender for the privilege of using the money is always quoted on the face of the note and agreed to by the signer. The principal times the rate, times the length of time the money is kept, will be the amount of interest that must be paid in addition to the principal on the date it is due or the date of maturity.

Simple interest is collected at specified times. Compound interest is not collected, but is added to the principal as in savings accounts. It is really interest drawing on interest, and so the account continues to grow.

MODERATOR: How can cashiers always be sure the money will be repaid, Sandra?

SANDRA: They aren't, and sometimes have difficult times collecting the money. To save themselves, however, unless they consider the borrower a "good risk," they require collateral with his signature. That means he must agree that something he owns, such as property, a car, bonds, etc., can be collected by the lender if the money is not repaid. A mortgage is about the same thing. It is security for the lender and makes the borrower realize the importance of repaying his loan when it's due.

MODERATOR: What happens if a person fails to make payments or to pay his other bills, Pat?

PAT: He loses his credit standing, which directly lowers his reputation in that community. This may follow him if he moves to another place, too, so it pays to make an effort to keep all bills paid.

MODERATOR: Where are the money and valuable papers of the bank kept, especially at night, Juanita?

JUANITA: They are kept in a large vault. The door weighs five or more tons and is set with bolts and time locks which allow the door to be opened only when the time has arrived as set on the locks. The vault walls are about eighteen inches thick and made of meshed wire and cement so they are almost impossible to drill through. The reserve money and valuable forms and records are kept there. Also, a section of it is reserved for the safe-deposit boxes which are rented by the citizens of the community. Anything the renter cares to place there may be put in. In order to open it the renter and the banker must both have keys. This is an added safety device so that no one but the renter can use his box.

MODERATOR: Now let's have Karen, Larry, Barbara (H), Jeanne, Donna, and Walter give a quick résumé of what we have learned. Karen, will you begin? Then each of the others will follow you in order.

KAREN: Always write checks in ink.

LARRY: In signing all legal papers and checks, always use the same signature.

BARBARA: The signature on all legal papers should be in the lower right-hand corner.

JEANNE: Fill in all unused spaces of a check with wavy lines to avoid having someone change it.

DONNA: Do not erase on a check. If a mistake is made, destroy the check and write a new one.

WALTER: Keep checkbook stubs filled in, and carefully check when cancelled checks are returned.

KAREN: Make an effort to make payments when due, or go to your banker and discuss the possibilities for time extension.

LARRY: Never indorse a check until you are facing the person cashing it, to avoid the chance of losing the check and allowing someone else to cash it.

BARBARA: Always be courteous and show appreciation to the cashier and all bank employees.

JEANNE: Do not hoard your money!

Deposit it in banks so it may be kept in circulation. Then many can use it and gain benefits from it.

DONNA: Make use of the Christmas Club Fund at your bank. It will help you to have money to spend at Christmas time.

WALTER: Our money system is built on trust in the system as felt by all Americans. Keep that trust and spread it by using it and thereby enjoying the privileges of our country.

MODERATOR: By this discussion today we hope we have succeeded in telling you some of the facts we have learned. We hope, too, that we have reminded you of some of the banking privileges about which you may have become careless. I, in behalf of the members of this class, want to thank the local bankers and bank employees for their help and co-operation which have helped to make this unit a success.

ANNOUNCER: As the conclusion of this program, we want you to know that you've been listening to the eight-to-three mathematics class of Lincoln School at Shawano, Wisconsin. We have discussed facts we've learned in a recent banking unit.

PEG MCCABE
Lincoln School
Shawano, Wisconsin

EDITOR'S NOTE: The complete script of the radio program does not appear in this article. Because of its length, part of the original dialogue has been deleted. We feel that the selected parts describe a class activity which may be of interest to the reader.

MATHEMATICS AT CENTRAL HIGH SCHOOL, FLORENCE, ALABAMA

Mathematics at Central High School is, of necessity, highly practical in nature. Most of the students enrolled do not expect to continue any formal training after graduation. They will become the farmers, factory workers, grocers, and homemakers of the area.

Few of our students will enter the professions. But for those who choose occupations requiring additional mathematical training, we offer the traditional mathematics courses, i.e., two years of algebra and plane geometry. In the event it is necessary, trigonometry and solid geometry are offered to individual students who plan engineering careers.

Our main concern, however, is with students whose mathematical training terminates in the high school. With this fact in mind, we have included in our curriculum the following courses: Junior Business Training, General Mathematics in the Junior High School, and General Mathematics in the Senior High School. These courses are closely related to the vocational interests of the students and the vocational opportunities in this area.

In the Junior Business Training course students are prepared to meet some of the problems of the business world. It deals with such topics as banking, insurance, and taxes, of which every ordinary citizen needs some knowledge. Junior High School General Mathematics includes practical applications of mathematics to other fields, such as carpentry, engineering, farming, etc., as well as a study of economics as related to mathematics. The Senior High School General Mathematics course is based as much as possible upon the needs of the students as found through preliminary inventory tests. Since, in many cases, this is the student's last opportunity to become proficient in mathematical areas, its content is wide and varied, including almost any needed topic. It is usually more advanced in nature than Junior High School General Mathematics; however, some individual instruction is on a very elementary level, depending upon the need of the individual.

In many instances the mathematics students are able to be of assistance as advisors in other classes, such as in the core. This not only gives the students a situation in which to practice what they have learned, but they also come to appreciate

(Continued on page 590)

MATHEMATICAL MISCELLANEA

Edited by

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and

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AND WHATEVER YOU'VE GOT TO DO
I'D BE SO HAPPY TO BE DOING IT WITH YOU

IN A mathematical way, that is. Just a different way of saying please send in your pet items. We sort of run in cycles, with the December issue going to press before we have seen the October issue in print, so it is hard to maintain continuity of topic. It will be some time before your contribution appears in print unless we can plan ahead as we managed to do on the following.

90. On Cyclic Sets of Digits

EDITOR'S NOTE: At our request Norman Anning, Alhambra, California, has sent in this note on cyclic sets of digits. The problem was mentioned in THE MATHEMATICS TEACHER, XLVI (October, 1953), page 433.

Any student who has put the reciprocals of 2, 3, 4, . . . 20 into the equivalent decimal fractions will understand what is meant by "periodic." He will be ready to admit that it is likely that every rational number can be expressed as a decimal fraction which either terminates or, from some point on, is periodic and never terminates.

Let us suppose he has been asked to observe (or better, to verify) these:

$$\frac{1}{7}=\frac{142857}{999999}=0.\dot{1}42857\dot{1},$$

$$\frac{2}{7}=\frac{285714}{999999}=0.\dot{2}85714\dot{2},$$

$$\frac{3}{7}=\frac{428571}{999999}=0.\dot{4}28571\dot{4},$$

and three others. He will grasp the meaning of "cyclic" and will be ready to see what this note is about. For instance, from

$$(1)(6)=(2)(3) \text{ it follows that}$$

$$(142857)(857142)=(285714)(428571).$$

Some may find such relations amusing but there is no compulsion. *De gustibus non disputandum.*

Suppose a person with time and patience notices that $99999=9\times41\times271$ and decides to look for interesting number combinations among the decimal expressions for $(j/271)$ where $j=1, 2, 3, \ldots 269, 270$. He will find that each division yields the decimal equivalents for five of these fractions. Thus,

$$\begin{array}{r} 0.17343 \\ \hline 271)\overline{470} \\ \quad 271 \\ \hline \quad 1990 \\ \quad 1897 \\ \hline \quad 930 \\ \quad 813 \\ \hline \quad 1170 \\ \quad 1084 \\ \hline \quad 860 \\ \quad 813 \\ \hline \quad 47 \end{array}$$

} and repeat.

This yields the following

j	$\frac{j}{271}$
47	$0.\dot{1}7343\dot{8}$
199	$0.\dot{7}3431\dot{7}$
93	$0.\dot{3}4317\dot{3}$
117	$0.\dot{4}3173\dot{4}$
86	$0.\dot{3}1734\dot{1}$

but it does not yield anything that is entertaining. To tabulate results for the possible values of j requires $270 \div 5 = 54$ similar divisions.

One who looks at the cyclic arrangement

$$\begin{array}{ccc} & 9 & \\ 4 & & 4 \\ & 4 \quad 6 & \end{array}$$

would not have any reason for suspecting that

$$\frac{4\,4\,6\,4\,9}{6\,4\,9\,4\,4} = \frac{6\,4\,9\,4\,4}{9\,4\,4\,6\,4}.$$

There is a reason. If we study the fractions

$$(a/271) = (b/99999)$$

we find the following

a	b
121	44649
126	46494
176	64944
134	49446
256	94464

In short, $b = 369a$. The proportion stated above follows from the accidental fact that 121, 176, 256 are associated with the same five digits in the same cyclic order in the column of b's. Finding only one such set in 54 justifies the use of "accidental." We have

$$\frac{1\,1}{1\,6} = \frac{1\,2\,1}{1\,7\,6} = \frac{1\,7\,6}{2\,5\,6} = \frac{4\,4\,6\,4\,9}{6\,4\,9\,4\,4} = \frac{6\,4\,9\,4\,4}{9\,4\,4\,6\,4}.$$

There is no rule for finding such relations any more than there is a rule for finding agates or truffles. Go where they are likely to be. If the student will put 9, 17, 24, 21, 64, 56, 49, 52 *seventy-thirds* into periodic decimal fractions, he will be able to find six or more relations like the one we have been looking at. Any denominator may be used; one which is a factor of a short string of 9's is to be preferred. For instance,

37, 41, 137, 239, 271, 333667.

The result is ephemeral and useful only for entertainment. I believe all the theorems involved are in most good texts in Number Theory. Gustav Wertheim in *Anfangsgrunde der Zahlenlehre* quotes six pages from Gauss (*Werke*, Volume I, page 382, and Volume II, page 412) and mentions in a note that the length of period for all primes under 100,000 is in print. That was the state of things in 1902. It is hardly to be expected that I would be able to add anything fifty years later except "for entertainment."

NORMAN ANNING
Alhambra, California

91. On the Nonexistence of Integral Roots

EDITOR'S NOTE: Louis L. Pennisi of the University of Illinois sends in the following criterion for nonexistence of integral roots. To our knowledge it is original, but we have learned from bitter experience that discoveries in mathematics are frequently rediscoveries. We would be interested in hearing from our readers on this question. At least the result was not known to several college teachers we questioned, and the proof is so simple that we can easily believe Mr. Pennisi's statement that "I have used this method in my classroom and the students have reacted very favorably."

Suppose that we are given a polynomial with integral coefficients

$$(1)\quad f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

We would like to have a very simple criterion for determining when $f(x) = 0$ has no integral roots. The answer is given by the following

THEOREM: *$f(x) = 0$ cannot have integral roots if both a_0 and $a_n + a_{n-1} + \cdots + a_1 + a_0$ are odd numbers.*

PROOF. The assertion that a_0 and $a_n + a_{n-1} + \cdots + a_1 + a_0$ are odd numbers is equivalent to saying that $f(0)$ and $f(1)$ are odd numbers. The proposed integral roots of $f(x) = 0$ are given by $x = r + 2k$, where $k = \cdots -2, -1, 0, 1, 2 \cdots$, and $r = 0, 1$. Thus,

$$\begin{aligned} f(x) = f(r+2k) &= a_n(r+2k)^n \\ &\quad + a_{n-1}(r+2k)^{n-1} + \cdots \\ &\quad + a_1(r+2k) + a_0 \\ &= a_n r^n + a_{n-1} r^{n-1} + \cdots \\ &\quad + a_1 r + a_0 + 2P(r, k), \end{aligned}$$

where $P(r, k)$ is a polynomial in r and k with integral coefficients and hence an integer. But $f(r)$ is an odd number whereas $2 \cdot P(r, k)$ is even. Hence $f(r+2k) \neq 0$. Thus $f(x) \neq 0$ for x any integer.

Louis L. Pennisi
University of Illinois

92. Theme Paper, a Ruler, and the Parabola

In this note we describe and illustrate some simple graphical layouts which yield points on a parabola. Two instruments are used. One is a ruler (graduations are convenient, and will be assumed, but are not necessary). The other is composition paper, or more precisely, the set of equally spaced parallel rulings on such paper. The diagram is made directly on the ruled paper, and is concerned only with points which lie on the rulings. The ruler is used to draw lines which cut or meet these rulings.

A diagram like Figure 1 is obtained as follows: Draw a line-segment $A_0A'_0$ traversing an even number of spaces, thereby making certain that its midpoint O is determined. Designate as l_0 the ruling which bisects $A_0A'_0$, and as l the ruling which bears point A_0. Working from l_0 toward l mark successive rulings l_1, l_2, $l_3 \cdots$ as shown. On l mark points A_1, A_2, $A_3 \cdots$ at equal intervals of suitable length s. Points on the parabola are now readily

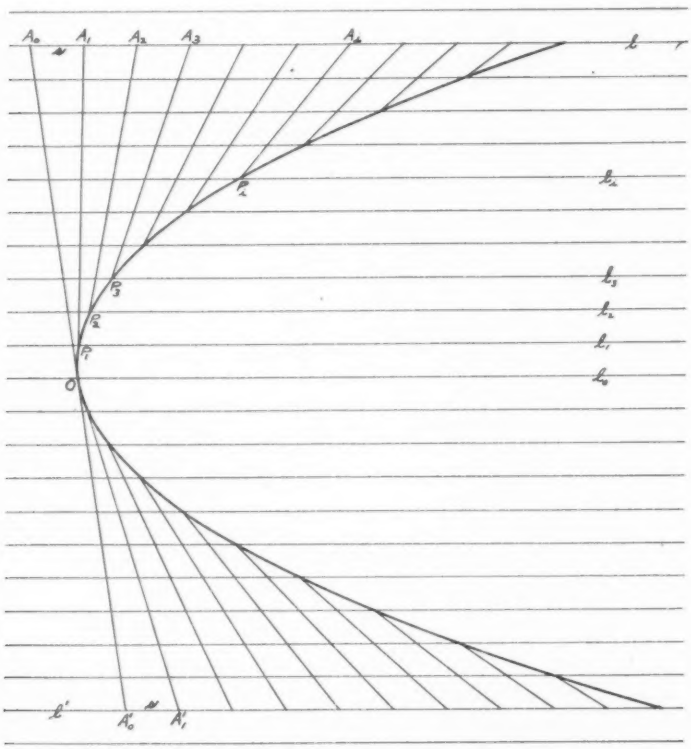

Fig. 1

obtained. P_1 is the intersection of l_1 and A_1O; P_2 the intersection of l_2 and A_2O; etc. In general, the intersection P_i of l_i and A_iO is a point on the curve. Point O is included in the set P_i, of course, and corresponds to $i=O$; it is unique in being a point of tangency. More points on the same parabola can be determined by laying off equal intervals s on l', and proceeding as above.

The method of the preceding paragraph has the advantage of giving a curve as large as the work sheet permits. It is obvious, however, that since O is the midpoint of $A_0A'_0$, the portion of the parabola determined by points on l' could have been obtained from l if the latter extended to the left of A_0, permitting equal intervals s to be marked in that direction. Figure 2 illustrates this, and also shows that the layout may be extended beyond the ruling l.

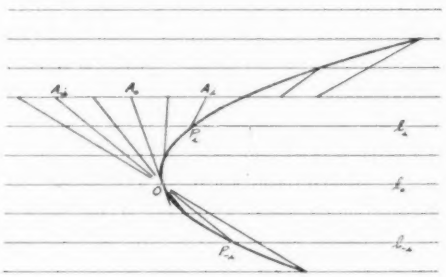

FIG. 2

Figure 3 shows why the curve produced by the procedure described is a parabola. The successive parallels l_1, l_2, l_3, $\cdots$, determine with the first two transversals OA_0 and OA_1 a series of similar triangles. From these it is evident that the intercepts on the parallels are proportional to 1, 2, 3, $\cdots$, as marked. Because of the equal intervals on some particular ruling l, there are equal intervals on each ruling. Now taking OA_0 as x-axis and l_0 as y-axis it is clear from the manner in which points on the curve are selected—one intercept on l_1, two on l_2, three on l_3, etc.,—that successive ordinates are proportional to 1^2, 2^2, 3^2 $\cdots$. Hence the selected points

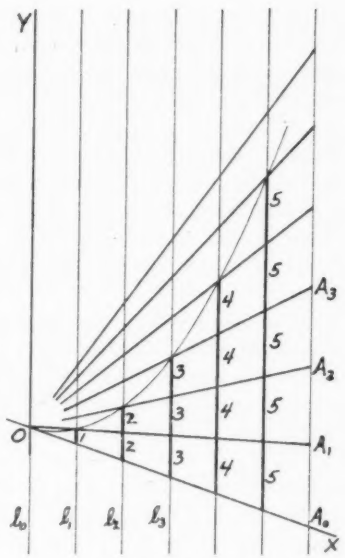

FIG. 3

lie on the curve $y=kx^2$. For both rectangular and oblique axes, then, the curve is a parabola.

The layout of Figure 4 does not require that equal intervals be marked on one or more rulings. It is assumed, however, that the sheet of paper has two straight parallel edges which cut the rulings. These edges are taken as l and l', and immediately

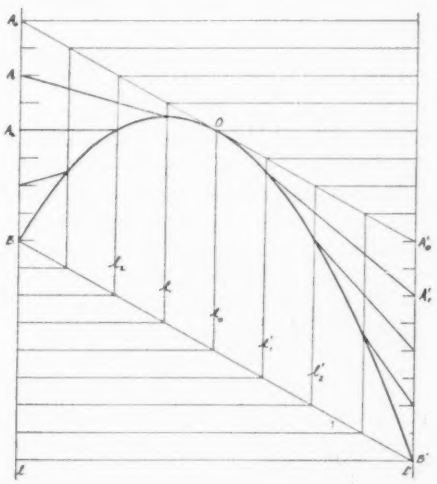

FIG. 4

furnish equal spaces because of the rulings. $A_0A'_0$ and BB' cross the sheet from edge to edge, and traverse the same even number of spaces. Hence they are parallel and are divided into the same even number of equal parts. Joining corresponding points of division on $A_0A'_0$ and BB' yields a set of equally spaced parallel lines, of which the middle one is taken as l_0. The procedure of Figure 1 is applied to these parallels.

For Figure 5 no assumptions are made other than the basic one—the work sheet carries parallel equally spaced rulings. $A_0BB'A'_0$ is a parallelogram obtained as follows: Draw A_0B to traverse an even number (say $2n$) of spaces. Draw $A_0A'_0$ to traverse $4n$ spaces. Then A'_0B traverses $2n$ spaces. Mark the midpoint C of A'_0B. Determine B' so that C is also the midpoint of A_0B'. Draw A'_0B' and BB'. The rulings on the work sheet divide $A_0A'_0$ and BB' into the same even number of equal parts. Join corresponding points of division. The result is a set of equally spaced parallel lines, the middle one to be taken as l_0. The work sheet rulings also divide A_0B and A'_0B' into equal parts. Take these lines as l and l'. Proceed as in Figure 1.

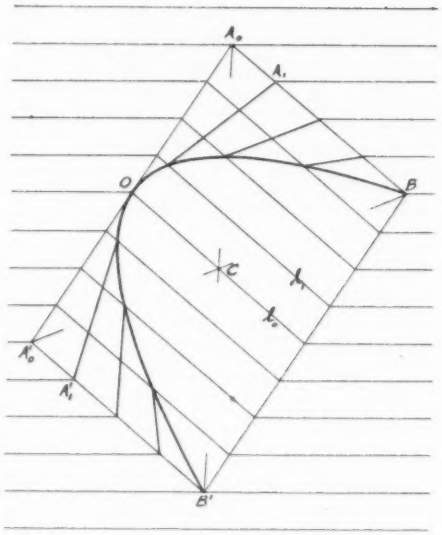

FIG. 5

Methods by which the hyperbola and the ellipse can be drawn with a ruler on composition paper will be shown in future issues.

ADRIAN STRUYK
Clifton High School
Clifton, New Jersey

What Is Going on in Your School?

(*Continued from page 585*)

the importance of mathematics outside of the classroom and in a real-life situation.

In all courses a major emphasis is placed upon enjoyment. Mathematics is made interesting to students through varied bulletin boards, puzzles, curve stitching, painting, wood carving, three-dimensional graphs, and a study of the history of mathematics. The present enrollment shows a marked increase in the number of students enrolled in mathematics courses, partially due to the fact that we are arousing natural interest.

Central High School uses the Satisfactory and Unsatisfactory system of grading. Satisfactory work is attained when the individual is working up to his own ability. Each week, individual conferences are held with students so that they may be conscious of their own progress or lack of it. Suggestions are constantly given the slow learner to improve his techniques of study. The fast learner is directed into related areas of interest when he has sufficiently mastered the subject at hand. It is hoped that each individual is brought to his own level of understanding and development.

MRS. JEAN PARKER
Central High School
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REFERENCES FOR MATHEMATICS TEACHERS

Edited by WILLIAM L. SCHAAF
Department of Education, Brooklyn College, Brooklyn, New York

The Miracle of Arithmetic

THERE IS little in this world to which number cannot be applied. Yet number exists only in the mind of man. Things of themselves are not countable—it is man who decides to count them. He can count them only after he has learned to count, not before. In short, the physical world does not impose a number system upon us. Man has created a logical abstract system —arithmetic—a veritable triumph of the mind. Casually taking his creation for granted, he passes it on to succeeding generations as a matter of course. With this magnificent heritage, as yet unborn multitudes will be able to understand the world about them far better than they otherwise could.

In structure, arithmetic is a closely knit system of ideas, constituting a self-contained and unified whole. Such a system cannot be learned piecemeal, nor by dealing only in specifics. Numbers, operations, processes, and relationships take on full meaning only as they become generalized. Arithmetic must therefore be taught and learned as a system.

As with all abstract ideas and generalized concepts, understanding is not achieved at once; it must of necessity emerge through experience. Such evolution and growth requires time. The process cannot be hastened unduly. Moreover, the learner cannot be expected to rediscover by himself that which took the race many centuries of laborious effort to invent and bring to perfection. Instead, we who teach must skilfully guide him in these unique ways of thinking—ways which are possible, in some degree at least, to all persons.

To bring children, in the brief span of a few years, to a point where they understand this system well enough to use it effectively and with confidence is a major accomplishment in which we may well take pride. It is a formidable task—indeed, almost a miracle. But it is also a very worthy task, for in the world of today, arithmetic literacy for the masses is more urgent than it has ever been. We can well understand what the great mathematician Lagrange meant, when, summoned before the tribunal during the French Revolution and asked what useful thing he could do to deserve to live, he humbly answered: "I will teach arithmetic!"

The following notes bring together data on several related aspects of arithmetic, to wit, numerals, numeration, number mysticism, and numerology. The reason is twofold: (1) because of their intrinsic interest and significance, and (2) because the literature on these subjects, although quite extensive, is unfortunately rather scattered.

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RESEARCH IN MATHEMATICS EDUCATION

Edited by KENNETH E. BROWN
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The Question: What research has there been in mathematics education in 1952?

The Study: A co-operative project by the National Council of Teachers of Mathematics and the U.S. Office of Education. Summary prepared by Kenneth E. Brown, Office of Education, Washington, D.C.

Fifty-seven research studies in mathematics education were reported for 1952. Fifteen of these studies were concerned primarily with the teaching of arithmetic in the elementary school. Five dealt with the teaching of algebra, two with geometry, and one in each of the fields of trigonometry and general mathematics. Further analysis of the research indicates that twenty-one of the studies were concerned with the content of certain courses in mathematics, twenty were studies of methods of teaching, seven were in evaluation, and five centered around problems in administration and organization of schools.

The experimental method was used as a method of research four times as frequently as the historical approach. In five of the studies the historical approach was used, and in twenty the experimental method. An emphasis on teaching for meaning is in much of the research. Eighteen studies included discussions and suggestions for making the teaching of mathematics more meaningful. Eight studies recommended various physical aids that may be used by the teacher and pupil to increase understanding.

Six of the research studies centered around the mathematics needs of special groups such as mathematics needed by engineers, by students in aeronautics, and by workers in vocational agriculture. Two of the studies attempted to answer the question, "What statistics should be taught in the secondary schools?"

Some of the conclusions found in the fifty-seven studies that were reported in 1952 are listed below. The number in the parentheses refers to the study in the bibliography upon which the statement is based.

The exclusive or prolonged use of ruler and compasses in modern high school plane geometry is not justified. (1)

The facts of research seem to imply that even though we offer different courses (algebra-general mathematics) the end product is the same. (2)

Segregation of slow learners for classroom work and their placement in smaller-than-usual groups with understanding teachers have made for progress in most cases in the basic skills of reading and arithmetic. (11)

The use of homemade materials (visual aids) is feasible and effective. (12)

There is a definite trend toward increasing the use of mathematics in research laboratories. (17)

The abacus is helpful in teaching the meaning of place value. (23)

Both I.Q. scores and seventh-grade marks are needed in predicting success in eighth-grade mathematics. (26)

The claim that the study of solid geometry will improve space perception abilities has little statistical backing. (40)

Of all students who take college algebra, 35 per cent must first take a remedial course. (56)

Summaries of these fifty-seven research studies were part of a joint project of the National Council of Teachers of Mathematics and the Office of Education. The summaries include author, title of study, year study was completed, source from which a copy of the study may be obtained, problem studied, procedures, findings, and conclusions. The titles of the

studies are listed in the following bibliography. Summaries of the research studies entitled "Mathematics Education Research Studies—1952," Circular No. 377 (free), may be secured from the Publications Section, Office of Education, Washington 25, D.C. If research studies in mathematics education completed in 1952 have been omitted, please notify the writer of this column, and they will be included in a supplement.

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AIDS TO TEACHING

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BOOKLETS

B. 175—Education for the Talented in Mathematics and Science

Superintendent of Documents, U.S. Government Printing Office, Washington, D.C.

Booklet; 6″×9″; 34 pages; $.15 each.

Description: This booklet is a report of a joint conference of the Committee on the Teaching of Science and Mathematics of the American Association for the Advancement of Science and the U.S. Office of Education held in November, 1952. The report summarizes and interprets the papers presented to the conference and the group discussions. It contains chapters on "Who Are the Talented in Mathematics and Science," "The Need for the Talented in Science and Industry," "Ways of Identifying the Talented," and "Providing for the Talented in Science and Mathematics." It discusses a great variety of methods and materials that are being used in schools throughout the nation to meet the needs of the gifted. An extensive bibliography is included.

Appraisal: This booklet reports practices that any teacher of science or mathematics can use to meet the needs of his gifted students. Although the practices are not reported in great detail, they are adequate to suggest a variety of techniques for the teacher, administrator, state department, professional organization, citizens and teacher-training institution to use depending upon the local situation. Providing for the gifted has become an increasingly critical need. Get this inexpensive pamphlet so that you will be more effective in challenging your superior students.

B. 176—Informal Problems for Ninth Grade Mathematics

Curriculum Bulletin, University of Oregon, Eugene, Ore.

Booklet; 8½″×11″; 7 pages; $.20 each.

Description: This booklet is a collection of twelve informal problems brought to class by pupils of a ninth-grade general mathematics class. An experimental class devoted one day each week to discussing these problems. The booklet includes a description of how the class arrived at the solutions to some of the problems.

Appraisal: The problems included in this booklet are of the typical puzzle variety found in the many books on recreational mathematics now available. Although the brief description of class activities is of some value, it would seem more advisable for the mathematics teacher to have the original sources of these problems available.

B. 177—Money: A Means of Exchange

Curriculum Bulletin, University of Oregon, Eugene, Ore.

Booklet; 8½″×11″; 15 pages; $.20 each.

Description: This is a resource unit centered around the development of money as a means of exchange. It includes an overview of the unit, the development of

the unit, the content of the unit, and suggested student activities. References include booklets and films. The content of the unit is primarily the history of money from barter to modern currency. Suggested student activities include questions to be answered, reports, projects, exhibits, sketching, and excursions.

Appraisal: This unit is written for use in grades six, seven, or eight for a period of three to five weeks. It is adequate in its outline of the history of money but is very brief in its treatment of the major aspect of a resource unit, namely, the use of a variety of materials and pupil activities.

EQUIPMENT

E. 134—KRECT Fraction Game

Self-teaching Flashers, 4402 S. 54th St., Lincoln 6, Nebraska.

Set of 60 cards; each $2\frac{1}{2}'' \times 4''$; $1.00 per set; 1952.

Description: The 60 cards are composed of 4 groups of 15 cards each. One group has a common fraction indicated on each card ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, etc. for 5ths, 6ths, and 8ths). Another group has a corresponding division indicated on each card ($1 \div 2$, $1 \div 3$, $2 \div 3$, $1 \div 4$, $3 \div 4$, etc.). The third group has a corresponding decimal fraction indicated on each card (.50, etc.). The last group has a corresponding percent indicated on each card (50%, etc.). Three ways of playing the game of KRECT are suggested: one is a variant of "Authors"; another, similar to "Pit"; and the third, a form of "Rummy." In each instance a player seeks to form "books" of 4 cards having the same "value." Scoring and determination of winner varies with each form of the game.

Appraisal: This drill device has one feature not found commonly in other somewhat comparable drill games with "fraction-decimal-percent" equivalents. Reference is made to the set of 15 indicated divisions ($1 \div 2$, etc.). Despite this fact, one claim made for the game is wholly unfounded and grossly misleading: "KRECT makes it easy to *understand* the relation between fractions, percents and hundredths, which so many find hard to grasp" (Italics added). The use of this game does not necessarily engender any *understanding of mathematical relationships*, nor is a child's success in playing the game necessarily in direct proportion to his degree of understanding. It is both possible and probable that through playing the game many children simply would improve their ability to make, more or less by rote, certain associations which are relatively meaningless and arbitrary. It is high time that we recognize games such as this for what they really are—drill devices that may have some use for a rightful purpose *following* the development of understanding and relationship—and stop believing them to be easy roads to certain other instructional outcomes which can be realized effectively only through other means and procedures. (Reviewed by J. Fred Weaver, Boston University.)

E. 135—Hundred-Bead Number Frame

The John C. Winston Company, 1010 Arch Street, Philadelphia 7, Pennsylvania.

Number frame; $20'' \times 30''$; 10 rods with 10 beads on each; Cohere-o-graph shield; $27.50.

Description: This number frame, which is 20 inches by 30 inches, is mounted on two legs and can be adjusted in height to meet the needs of the class. The frame has ten wire rods with ten $1\frac{1}{4}''$ beads on each rod. Five of the beads on each rod are red and five are yellow. The cohere-o-graph shield slides in front of the rods so that different numbers of beads can be hidden from sight as desired. The cohere-o-graph shield also serves as a display surface for pictures, symbols, or vocabulary to be related to the number relations illustrated by the beads. A shelf on which supplementary materials may be placed extends across the front of the frame.

Appraisal: This number frame is similar to a large abacus and thus lends itself to a

variety of uses for illustrating, visualizing, discovering the meaning of numbers, number combinations, decimals, and percent. Beads of two colors are used for the purpose of reducing the need for counting. The simultaneous use of the cohere-o-graph shield should facilitate the transfer from concrete representation to the verbal or symbolic. This gradual transfer from concrete to pictorial to verbal to symbolic is often overlooked in the use of instructional material. The size of this device and the possibility of shielding beads not being used should be features that meet classroom needs.

E. 136—Fruit Plate

Playskool Manufacturing Co. (Available through school supply houses), 1750 North Lawndale Avenue, Chicago 47, Illinois.

Models; wood; diameter 8″; plate, apple, pear, orange; $2.00.

Description: The purpose of this fruit plate is to demonstrate the meaning of fractions and the addition or subtraction of fractions with three dimensional models. The device consists of models of fruit and a plate. The fruit is shaped and painted to represent an apple, a pear, and an orange. The pear is divided into 2 halves, the apple into 3 thirds, and the orange into 4 fourths.

Appraisal: These large models beautifully finished and lacquered will provide attractive, clean, odorless models for classroom demonstrations. It should be worth the price to be free of the stains and mess connected with the use of real fruit and to have the convenience of having the models available whenever needed.

E. 137—Plain Cubes

Ideal School Supply Company, 8322 Birkhoff Ave., Chicago, Illinois.

Models; Cubes 1″×1″×1″; $1.50 per 100 plain; $1.80 per 100 colored.

Description: These one-inch wooden cubes are available plain or in six standard colors. They may be used to teach the meaning of numbers, number combinations, fractions, form, or design or volume. Every arithmetic and mathematics teacher should have an adequate supply of these cubes for student manipulation.

E. 138—Addo

Kenworthy Educational Service, Inc., Buffalo, N. Y.

Game; 36 game cards, 100 calling coins, 100 deposit coins; $1.50.

Description: The purpose of this game is to provide drill on the addition combinations in a game situation. Each child participating is given a deposit card which contains nine numbers. The nine numbers on each card are a different combination of numbers from 0 through 18. Small round calling cards contain all the addition combinations up to 9+9=18. These calling cards are mixed up at random and the child selected as a banker draws them one by one from a box. He then calls out the combination drawn without giving the sum. The players must know the answer so that they can find the sum of the combination called on their deposit card. If the sum appears on their card, they cover it with a small round blue card called deposit coins. The first player to cover all the numbers on his deposit card calls out "Addo," and, if his answers are correct, he is the winner.

Appraisal: This simple game should help to break the monotony of drill on addition combinations. The randomness of answers on the deposit card should help the slower child win more frequently than he would from direct competition.

E. 139—Grove—Yoder Abacus

Yoder Instruments, East Palestine, Ohio.

Demonstration device; $8\frac{1}{2}$″×30″×$1\frac{3}{4}$″; $10.00.

Description: A white case, with two adjustable legs to support it, has three vertical wires running the whole length. Each wire has ten roundish beads, about $1\frac{1}{4}$″×$1\frac{1}{4}$″, some red, some blue, and some

yellow. There are springs in each bead so that it will stay at any point on the wire. It is possible to take any one of the sets of ten beads and its wire out of the frame to simplify the set-up to twenty, or even ten, beads but it is not possible to remove or change beads from a wire.

Appraisal: This is a large, simple, attractive, and useful abacus. It has vertical bead wires which give place value similar to place value in numbers to distinguish it from other commercial or home-made abacuses. It is very well made and should be very sturdy. It is not as simple as the single wire abacuses or as complicated as the abacounter (*E. 67*, reviewed March 1952). For elementary grades this should be an extremely useful piece of equipment.

E. 140—"Wrico" Geometry Guide #1

Spaulding Moss Co., Boston, Mass.

Stencil; 5″×6″; $.35.

Description: A stencil of clear plastic in which sixteen of the most common geometric figures, including triangles, quadrilaterals, regular polygons, and circular sections, are cut. Each figure is labeled on the stencil. On the flap of the envelope a condensed definition of each is given. There are also definitions of a few other figures which are not shown.

Appraisal: This could be a handy piece of equipment for each individual in a high school geometry class. It also could be used for arithmetic and algebra. The figures are a good size for making neat diagrams and are sufficiently large for marking. At thirty-five cents each, these stencils are not a bargain, but in quantity they might be obtained a little cheaper. (Reviewed by George G. O'Brien, Boston University)

FILM

F. 98—Insurance Against Fire Losses

Encyclopaedia Britannica Films, Inc., 1125 Central Ave., Wilmette, Illinois.

Color; 16 mm.; ($140 [$5.50 for 1–3 days use]); 1½ reels, 15 min.; sound.

Description: In this film we hear about the problems involved if one loses his house by fire; there is some discussion of the causes of fire and of the advantages of sharing the loss. Through the explanation, beginning with a probable $10,000 loss for each $10,000,000 involved, the idea of "Fire Insurance" is developed. Then the factors involving risk—neighborhood, adjacent types of properties, type of construction, use of the property—are mentioned.

Appraisal: The fact that this film is in color adds greatly to the attention, but the color has little functional purpose and therefore is mostly decoration and attention-getting. There is a slight bit of mathematics mentioned and explained in showing the connection with Rates of Insurance and the Yearly Premium.

The cartoon technique used, mostly a static series of still pictures, is unfortunate and not especially interesting or entertaining after a short period of time. The subject of "Insurance Against Fire Losses" may be explained by other media than films. It is not one of the best subjects to be used for film treatment.

F. 99—The Number System

Encyclopaedia Britannica Films, Inc., 1125 Central Ave., Wilmette, Illinois.

Color; 16 mm.; sound; 1953.

Description: A boy counting his toy soldiers sets the stage for counting by tens. This scene is brought to life by pictures of Boy Scouts marching in groups on a field. A chart divided into two sections, one section for tens and one for units, is used to record the number of scouts. The scouts march in groups of ten and report as tens to show the number of tens in 25, 44, and 59. The number of units is reported by each member of the group which is less than ten counting off. Changing from units to tens is shown by having a late scout join the 59 scouts to make 60. The film ends by asking the audience questions about the number of tens and ones illustrated by groups of toy soldiers.

Appraisal: This film should add interest and meaning to numbers by using groups of boys to illustrate place value. Its title is somewhat misleading since it does not deal with the number system other than to show the meaning of figures in the ten's and unit's place. However, it is to be commended in its presentation of a single concept. The photography is simple and realistic. It should suggest to the teacher how objects or pupils can be grouped and recorded to illustrate place value.

F. 100—How To Judge Authorities

Coronet Instructional Films, Coronet Building, Chicago 1, Illinois.

B&W ($50) or Color ($100); 16 mm.; 400 ft., 11 min.; sound; 1948.

Description: Bill is considering law as a profession. He has consulted a successful and enthusiastic attorney and has read a book concerning the legal profession. Bill is confused because there is a conflict between statements of "authorities." Mr. Penner, author of the book, describes the "legal profession as dull and tiresome"—"years of drudgery"—"lawyers are often unhappy and unsuccessful." Mr. Morley, the attorney, is so inspirational that Bill would like to be an attorney such as Mr. Morley. How is he to decide?

Bill's teacher suggests that he investigate both sources under headings (1) Internal Evidence, (2) Author's Experience, and (3) Bill's Own Experience. Bill lists points under each in true debate form. In the end the question is left to the student to reach a sound, intelligent decision.

Appraisal: This film was shown to a class in Plane Geometry. It pointed out the conflicts one finds in radio, lectures, books, magazines, and television today. Young readers like to be discriminating readers and for this reason they liked this film. I used the film to encourage geometry readers to be "doubting Thomases" and not accept every printed and oral word. I believe "How To Judge Authorities" serves a real purpose in teaching geometry in its approach to clear thinking. (Reviewed by ETHEL CAIN, Des Moines, Iowa.)

F. 101—How To Judge Facts

Coronet Instructional Films, Coronet Building, Chicago 1, Illinois.

B&W ($50) or Color ($100); 16 mm.; 400 ft., 11 min.; sound; 1948.

Description: This film pictures the story of Jim, a high school Sophomore, writing a sensational story for the school paper. He wanted big headlines to read "Football Money Goes For Movie Projector." When questioned as to facts in his story, he knew they were reliable because he had interviewed a football player, track man, a mother, and a movie operator—the only other person the editor had asked him to see was the high school principal. When he finally talked with the principal, the principal gave him the facts, squelched the false rumors, and the story printed was far from Jim's original.

Appraisal: My Plane Geometry class enjoyed the film because it was entertaining as well as instructional. It pointed out very clearly the five errors made in gaining facts for this original story:

1. False analogy
2. Irrelevant facts
3. False assumptions
4. Platitudes
5. Definitions not clearly given (double meaning)

It gave a good send-off for a class assignment the following day on examples of faulty reasoning. (Reviewed by ETHEL CAIN, Des Moines, Iowa.)

FILMSTRIPS

*FS. 177 through FS. 178—Light on Mathematics—Algebra**

FS. 177—Positive and Negative Numbers (60 frames)

FS. 178—Ratios and Proportions (50 frames)

* See also *FS. 3*, Problem Analysis, *FS. 64*, Exponents and Logarithms, *FS. 69*, Arithmetic of Algebra and *FS. 70*, Equations and Formulas; April 1948, February 1950 and May 1950.

Jam Handy Company, 2900 E. Grand Boulevard, Detroit 11, Michigan.

B & W ($4.00 each).

Description of FS. 177: The meaning of positive and negative numbers is illustrated by counting on a number scale. However, the rules for subtraction are simply stated and then applied to specific combinations. The rules for multiplication and division are related to nonmathematical situations involving friends, enemies, wins and losses, and to repeated additions or subtractions.

Appraisal of FS. 177: Many more applications of opposite quantities could have been given to supplement textbook treatment. Emphasis on the meaning of processes rather than on the following of a stated rule would have made this filmstrip more valuable. As it is, it repeats textbook treatment.

Description of FS. 178: Although there are many ways of comparing things such as by the sense of touch or sight, a quantitative method of comparison is through the use of a ratio. To solve a specific problem, for example, the relationship between a bicycle's chain gears and the speed of the gears, a proportion must be used. According to this strip the solution of these proportions is accomplished by representing the unknown by x and then cross-multiplying. Illustrations of direct proportions include a quantitative comparison of the changing volume of a balloon under different temperature conditions and the descriptive relationship between perspiration and the amount of work performed by the worker. Other illustrations of direct and inverse proportions involve airplane servicing, airplane travel, map reading, and house building. A final example involves the solution of a compound proportion.

Appraisal of FS. 178: Some unique illustrations of the use of proportions are included in this film but most of them parallel textbook discussion. Emphasis on cross-multiplication as the method of solution is of questionable value. The comparison of yards to feet is done in a manner that is inaccurate.

*FS. 179 through FS. 183—Light on Mathematics—Geometry**

FS. 179—Addition and Subtraction in Geometry (56 frames)
FS. 180—Multiplication and Division in Geometry (54 frames)
FS. 181—Angular Measurement (76 frames)
FS. 182—Constructions (62 frames)
FS. 183—Scales and Models (83 frames)

Jam Handy Company, 2900 E. Grand Boulevard, Detroit 11, Michigan.

B & W ($4.00 each).

Description of FS. 179: Geometric terms such as lines, angles, areas, and volumes are illustrated by concrete objects. Arithmetic addition and subtraction are then related to the addition of lines, angles, areas, and volumes. These methods of addition and subtraction are used to discover many relationships between lines and angles such as the length of parallel lines between parallels, the equality of vertical angles, the equality of angles formed by parallel lines and a transversal, and the sum of the angles of a triangle. These five basic relationships between lines and angles are used to solve a variety of geometric problems.

Appraisal of FS. 179: This film covers such a wide variety of topics that its proper use is difficult to ascertain. It would seem more appropriate to have pupils make the discoveries illustrated than to have them shown in pictures.

Description of FS. 180: It is the purpose of this filmstrip to show that geometry applies multiplication and division to lines, surfaces, and solids. The meaning of area is illustrated by a group of equal pencils forming a rectangle. These pencils are then moved to illustrate that the area of a figure is dependent on the altitude and

* See also *FS. 65*, Trigonometry and *FS. 105*, Vectors; March 1950 and February 1952.

base rather than shape. Formulas for the areas and volumes of a variety of geometric figures and solids are illustrated without showing how the formulas were derived. The place of π in the formulas for the circumference and area of a circle and the volume of a sphere is discussed. A brief proof of the Pythagorean Theorem is included.

Appraisal of FS. 180: This filmstrip covers a variety of topics which sometimes lack coordination; for example, similarity follows a discussion of areas. Although the use of pencils to form a surface is unique and appropriate the greater part of the strip uses textbook techniques and drawings. Its failure to use meaningful language is illustrated by the caption for a cube, "any line cubed will give us a cube." Similar statements such as "shape does not affect the formula" fail to include the conditions under which this statement is true.

Description of FS. 181: Angles are measured according to four systems; i.e., degrees, radians, mils, and trigonometric functions. The advantages, method of measurement, and applications in each system are illustrated. Measurement in degrees is illustrated by the protractor, transit, sextant, and compass. The radian is shown as useful in problems concerned with revolutions or distance traveled. The mil is used in connection with military gunnery. A considerable part of the filmstrip is devoted to showing the definitions and relationships of the six trigonometric functions.

Appraisal of FS. 181: This filmstrip will be most appropriate in a unit on trigonometry. Except for the trigonometric functions, most frames picture applications of the angles involved.

Description of FS. 182: The constructions illustrated are limited to those which can be performed using a compass and a straight edge. Step-by-step drawings are given to show how to copy a line, angle, and arc. These methods are then used to show how to add, subtract, multiply, or divide lines, angles, and arcs. Additional constructions include the right triangle and the hexagon with a final picture suggesting an application in mechanical drawing.

Appraisal of FS. 182: This topic could well be presented in terms of its applications rather than in terms of textbook drawings. Instead of limiting its constructions to the methods of Euclid it could have illustrated the methods used by the draftsman.

Description of FS. 183: This filmstrip shows how drawings, blueprints and maps may be used to represent things by geometric projection and how models represent things in three dimensions. In the use of these drawings and models it is important that an appropriate scale be used and that the drawing or model represent all parts in this scale. The solution of a problem such as determining the height of a pole by measuring its shadow is used to illustrate how to use scale drawings. Pictures, orthographic projections, contour lines, and airplane models are used to illustrate a variety of applications of scales and models.

Appraisal of FS. 183: This filmstrip contains a collection of drawings that could be useful in introducing scale drawing. Much verbal material is included.

*FS. 184—Analytical Geometry**

Jam Handy Company, 2900 E. Grand Boulevard, Detroit 11, Michigan.

B & W ($4.00); 45 frames.

Description: The graph of a straight line is drawn by plotting two points or by the use of slope-intercept form. Conic sections are illustrated and related to these representative equations. The equation of each conic section is graphed and methods of drawing each curve illustrated. The simplification of quadratic equations by a translation of the axis completes the strip.

Appraisal: Analytical geometry is a

* See also *FS. 62*, Plotting Graphs and *FS. 66*, Graph Uses; February 1950 and March 1950.

topic that is too involved to be presented in 45 frames even when the discussion is limited to graphing straight lines and conic sections.

FS. 185—Pilot Problems (40 frames)

FS. 186—Dead Reckoning (85 frames)

Jam Handy Company, 2900 E. Grand Boulevard, Detroit 11, Michigan.

B & W ($4.00 each).

Description of FS. 185: This film approaches the basic problems of air navigation from the practical viewpoint of the pilot. The examples in this film are based on the explanations given in other films of this series. The problems involve the formula $S = dt$, course correction for magnetic variation, deviation and wind, scale representation of vector triangles, radius of action formula, orientation to the radio beam, and altimeter corrections.

Description of FS. 186: Planning a flight between two cities furnishes the setting for planning to navigate by dead reckoning. A variety of factors involving distance, direction, and position must be considered. These factors are related to map distortions, compass deviations, and wind vectors. Solving the navigational problems involves the statement of numerous rules and formulas and the drawing of courses and vector triangles.

Appraisal of FS. 185 and FS. 186: These filmstrips may be used to show interesting applications of mathematics. However, most of the content is verbal and thus resembles textbook presentation.

INSTRUMENTS

I. 41—Modern Computing Abacus

Ideal School Supply Company, 8322 Birkhoff Ave., Chicago 20, Illinois.

Device; wood base; 4 wire loops; counting discs on each wire; decimal insert; $3.00.

Description: This is a manipulative device to visualize place value in computing with whole numbers and decimals. It is similar to an ordinary abacus in that discs are manipulated on a wire to picture the meaning of four place numbers. It is different from an abacus in that the wires are vertically looped 180° at the middle with a panel between the two sections of wire. In this way only the discs on one side of the loop are visible. It has twenty discs on the units wire to facilitate showing carrying in addition and multiplication and regrouping in subtraction and division. A decimal point insert makes possible computation in decimals.

Appraisal: This computing abacus has several advantages over an ordinary abacus. The looping of the wire which hides unused discs and permits the use of the wires in vertical position gives a more realistic picture of place value than work on horizontal wires. The use of 18 discs on each wire makes it versatile in showing carrying and regrouping. It is also suitable to illustrate computation in other number bases.

MODELS

M. 31—Paper Slide Rule

William R. Ransom, 13 Barrows Road, Reading, Massachusetts.

Paper model; 8 inches long; 15 cents or 10 for $1.00

Description: Slide rule scales, *A*, *B*, *C*, *D*, *L*, *S*, and *T* are printed on a sheet of heavy bond paper. Simple directions for cutting and folding and a celluloid runner are included. The result of this cutting and folding is two strips of paper, one representing the body of the rule and the other the slide. Grooves are formed by folding so that the scales can be easily manipulated for computation. Directions for multiplication, division, and the solution of triangles are included.

Appraisal: A simple, inexpensive slide rule is obtained in which the student participates in its construction. Its accuracy and durability are limited, but often will be satisfactory for the short instructional time that is devoted to slide rule manipulation.

BOOK SECTION

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BOOKS RECEIVED

Elementary

Learning To Use Arithmetic 2, by Agnes G. Gunderson and George E. Hollister. D. C. Heath & Company, New York, 1953. Paper, 1+144 pages, $.76.

College

Elementary Mathematics From An Advanced Standpoint, by Felix Klein. Dover Publications, Inc., New York. Paper, iv+274 pages, $1.50.

A History of Astronomy from Thales to Kepler, by J. L. E. Dreyer. Dover Publications, Inc., New York, 1953. Cloth, iii+438 pages, $1.95.

Lectures on Analytic and Projective Geometry, by Dirk J. Struik. Addison-Wesley, Cambridge, 1953. Cloth, v+291 pages, $6.50.

Problem Book in the Theory of Functions, Volumes 1 and 2, by Konrad Knopp. Dover Publications, Inc., New York, 1948. Paper, v+126 and v+138 pages, $1.25.

Publications de L'Institut Mathematique, (Tome 1, 2, 3), Belgrade, 1947, 1948, 1950. Paper, viii+141, 8+269, and 2+306 pages, no price given.

A Refresher Course in Mathematics, by F. J. Camm. Emerson Books, Inc., New York, 1953. Cloth, 5+240 pages, $2.95.

Squaring The Circle, A History of the Problem, by E. W. Hobson. Chelsea Publishing Company, New York, 1953. Cloth, 1+110 pages, $3.25.

Miscellaneous

An Introduction to Symbolic Logic, by Susanne K. Langer. Dover Publications, Inc., New York, 1953. Cloth, 6+367 pages, $1.60.

Relativity and Reality, a Re-Interpretation of Anomalies Appearing in the Theories of Relativity, by E. G. Barter. Philosophical Library, New York, 1953. Cloth, ix+131 pages, $4.75.

Substance and Function and Einstein's Theory of Relativity, by Ernst Cassirer. Dover Publications, Inc., New York, 1953. Paper, iii+465 pages, $1.95.

The Works of Archimedes, by T. L. Heath. Dover Publications, Inc., New York, 1897. Cloth, vi+326 pages, $1.95.

REVIEWS

Analytic Geometry and Calculus, Lloyd L. Smail. New York, Appleton-Century-Crofts, Inc., 1953. Cloth, xlv+644 pp., $5.50.

The author says "this book is intended for a combined course in analytic geometry and calculus."

This purpose has been accomplished by interweaving materials of the two courses into a teachable sequence that should prove interesting to the student. After developing the basic concepts of analytic geometry up to the detailed study of the conics, the author presents the introduction of differential and integral calculus through the medium of algebraic functions. A brief development of the geometry of the conics is followed by the usual calculus topics. A study of solid analytic geometry precedes the work on partial derivatives and multiple integration.

The essentials of analytic geometry are well presented. The order of topics has been slightly modified, such as the early introduction of curve tracing. Polar co-ordinates and parametric equations are presented after the beginning work in the calculus so as to introduce the work of differentials with parametric equations. The normal-form of the equation of a straight line has not been included.

The text is not wordy and the student will find guidance in the one or more examples of specific exercises for each section. There are ample exercises and applications with answers included for the odd numbers. Starred topics are given for special interests.

The calculus materials are complete for a first course and have been tested in the previously published book by the author.

Anyone interested in teaching a combined course should examine this book. The author has done a thoughtful job of interweaving the essentials of analytic geometry with the basic elements of calculus. There is no indication that this course has been developed in the classroom.
—FRANCIS R. BROWN, Assistant Professor of Mathematics, Illinois State Normal University.

Articles on the History of Mathematics: A Bibliography of Articles Appearing in Five Periodicals, Cecil B. Read. The Municipal Univer-

sity of Wichita Bulletin, University Studies No. 26, 1952. Paper, 32 pp., $.50.

The purpose of this publication is to present a complete bibliography of articles on the history of mathematics which have been published, up to January 1, 1952, in five periodicals: *The American Mathematical Monthly*, THE MATHEMATICS TEACHER, *National Mathematics Magazine*, *Scripta Mathematica*, and *School Science and Mathematics*. The articles are listed by author in several convenient categories. This bibliography will serve a variety of useful purposes; in particular, it will be of great value to teachers and students of the history of mathematics, and it is a valuable source of suggestions for mathematics club programs and enrichment material.—H. D. LARSEN.

Beginning Algebra for College Students (Second ed.), Lloyd L. Lowenstein, Kent State University. New York, John Wiley and Sons, Inc., 1953. Cloth, xiii+279 pp., $3.50.

The first year of algebra or "algebra to quadratics" expresses the content of this textbook. The author has nicely designed this textbook to the mental maturity of the college student and organized it for use in the college classroom. Many colleges find it necessary to offer first-year algebra and this text is admirably designed to meet this need in a three- or five-hour course to cover the first year of high school algebra.

Some educators may doubt the author's implication that studying algebra is "racking one's brains" and that such activity will teach him "how to think." Further, some may doubt that "after studying this book the student should be qualified to enter a course in college algebra," unless it is a five-hour College Algebra course that includes the third half-year of algebra.

An examination of the textbook reveals the neat organization of the material, the conversational tone of presentation, the clear-cut objectives, and the casual yet rigorous inclusion of the foundation, laws, and rules of algebra. All this is done in a natural and meaningful manner. One cannot help but feel that this is an expression of the author's own effective and excellent teaching. The complete omission of the words and the use of "canceling" and "transposing" is highly commendable. The laws of algebra are not only included but included in such a manner that no teacher will skip over them lightly, and every student should develop a clear concept of their meanings and uses.

The exercises are plentiful, and seem to be nicely graded and well adapted to aid the student in transferring his knowledge of arithmetic to algebra and developing the needed concepts of algebra. The answers to the odd-numbered exercises are included at the back of the book.

The early introduction and continued use of the inequality signs deserves special note. The last two chapters on the Real Number System and the Complex Number System are somewhat disappointing as they are mainly operational. The Appendix should give the better students a better understanding of the questions raised throughout the text and possibly inspiration to further investigation.—CHARLES B. TUCKER, Kansas State Teachers College, Emporia, Kansas.

College Algebra, Ross H. Bardell and Abraham Spitzbart. Cambridge 42, Massachusetts, Addison-Wesley Publishing Company, Inc., 1953. Cloth, ix+197 pp., $3.50.

This book attempts to combine material in a small text for the beginning student with a limited background and for the student with one and one-half years of high school algebra. The first four chapters are essentially a review of the first year of high school algebra. A few of the topics included in older texts are purposely omitted and more emphasis is placed on the concept of function. Ample exercises seem to be provided through the text and three sections of review exercises at the ends of Chapters 4, 7, and 13 provide opportunities to evaluate the achievement of the student.

Good teaching in the classroom is needed in a few cases to keep the student from following rules and formulas in a mechanical manner rather than from an understanding point of view. An experienced teacher should be able to develop a strong course for those students with not more than one and one-half years of high school algebra by the use of this text. The print is clear and on good quality paper. The figures are neat and accurately drawn. This gives the book as a whole a neat, clear appearance.—OTHO M. RASMUSSEN, University of Denver, Denver, Colorado.

College Algebra and Plane Trigonometry, James H. Zant. Boston, Ginn and Co., 1953. x+418 pp., $4.00.

Zant's *College Algebra and Plane Trigonometry* covers in one text the material often taught in the freshman year of college mathematics. The proofs, wherever given, are of sufficient rigor for the students usually encountered on this level. Most of the problems are chosen carefully, though, on some topics, the number is hardly sufficient. This is especially true in the early part of the text, the trigonometric identities, and the written problems.

The solutions of the cubic and the quartic equations in terms of radicals would add much to the value of this book. It is adequate, however, for courses that require no more than one and one half units of algebra for entrance credit. —HERBERT L. LEE, University of Tennessee, Knoxville, Tennessee.

College Geometry, An Introduction to the Modern Geometry of the Triangle and the Circle, (second ed.), Nathan Altshiller-Court, New York, Barnes & Noble, Inc., 1952. Cloth, xix+309 pp., $4.00.

A new textbook in modern geometry is not exciting news. A textbook on this subject by Dr. Altshiller-Court is front-page news. Al-

though this book is supposed to be a "second edition," it bears resemblance to the first edition not much further than the title. The first edition enjoyed the reputation of a best seller, of a first-rate textbook, and the reviewer ventures to predict that the new book will outrank its predecessor.

If there can be a "seductive book" on geometry, the author succeeded in writing one. Were it not for dateline to be met, the reviewer would still be engrossed in reading this book. The presentation of the various topics is fascinating. The development of the subjects and the style are surprisingly clear and simple. As a textbook for college geometry classes it is an example of perfect writing. As a source of materials for high school teachers this work is encyclopedic.

There are very few elements on the debit side. The manner in which the historical and bibliographical notes are presented is unfortunate. These references would have been more effective if they were distributed throughout the book in the appropriate places. There should have been more uniformity in the drawing of the diagrams. The author should have employed a standard symbolism for primary, secondary, and subsidiary lines. He attempts this on a few occasions, but these are rare.

If another prediction is permissible, the reviewer suggests that this book will be a prized possession of the serious student of mathematics. This work is a real contribution to the literature on geometry.—Aaron Bakst.

Elementary Analysis, Kenneth O. May. New York, John Wiley & Sons, Inc., 1952. xviii +635 pp., $5.00.

Students who master the content of this book will have a very satisfactory equipment in first-year college mathematics. The book is packed—almost overstuffed—with good material. It is intended for students with a minimum preparation of one year each of high school algebra and plane geometry, but the author says that there is ample material for students with more extensive preparation, and this is by no means an overstatement.

Fourteen chapters of more or less integrated mathematics are given, including algebra, trigonometry, analytic geometry, and some calculus. One chapter each is given of logic, complex numbers, and solid analytics. There is some treatment of inequalities, determinants, permutations, and combinations, but none of probability. The derivative is introduced in Chapter 5 and used mildly in a number of connections from that point on. Differentiation formulas are often introduced without derivation, sometimes as a part of an exercise. The anti-derivative and definite integral are considered in Chapter 12, with too little integration with the rest of the chapter.

The notation and symbolism, although not difficult, are unnecessarily complicated. Even the numbering system in the book itself will be regarded by some readers as being rather clumsy. Thus, *6.6.3 means the third theorem in the sixth section of Chapter 6. Symbols and terms include: $\rightarrow$, $\leftarrow$, $+\!\!\rightarrow$, $\nrightarrow$, $\in$, $\notin$, $\subseteq$, $\cap$, null and disjoint sets, the sigma function, and "$X\in(A\cap B)\leftrightarrow X\in A$ *and* $X\in B$." Even the vinculum has not been entirely overlooked. We believe that symbols not actually needed in a textbook might well be left for introduction where their use is required.

Four tables are included, all of them four-place: powers and roots, common logarithms (actually mantissas), natural logarithms, natural values and logarithms of sines, cosines, tangents, and cotangents. Angles are given in 10′ intervals and corresponding radian values are included. No proportional parts are found in the tables, which your reviewer regrets, but others might prefer.

Exercises are sufficient in number and varied in degree of difficulty. They are set in rather small type, as, for that matter, is the entire book. But this was obviously necessary to keep the number of pages from going beyond bounds.

On the whole, this is a very creditable book, done by one who knows how to organize his work. To those who are looking for a solid treatment of first-year college mathematics in unified form, we recommend a consideration of this book.—F. W. Kokomoor, University of Florida, Gainesville.

Elements of Algebra for Colleges, Vaughn B. Caris. Boston, Ginn and Company, 1953. Cloth, vi +307 pages, $3.30.

This book is designed for college students who have had a course in high school algebra. The best part of the book is the review of the fundamentals of algebra which are developed in a language student can understand and are clearly illustrated with examples throughout their development. The remainder of the book is treated in a similar manner, but such treatment is not nearly as appropriate for the introduction of new ideas.

Like most mathematics texts it attempts to tell students about mathematics rather than help them understand the nature of mathematics. Since many students using such a text will not be taking additional courses in mathematics, it probably should have included introductions to statistics and indirect measurement. Conic sections and square root are presented in an interesting and teachable way. Although the exercises do not include many practical problems, they do give students practice in applying principles in a wide variety of situations. Teachers of second-year algebra either in college or high school should find this text valuable as a teaching aid.—Oscar F. Schaaf, University School, Ohio State University.

Fundamentals of College Mathematics, Richard E. Johnson, Neal H. McCoy, and Anne F. O'Neill. New York, Rinehart and Company, Inc., 1953. xiv +479 pp., $6.00.

This book assumes a thorough preparation before the student begins its use—three years or

more of mathematics in secondary school or college. It includes some work, one chapter of fifty-nine pages, on trigonometry, largely analytic, "a rather thorough introduction to analytic geometry and a brief account of some important topics from algebra." However, "the principal direction of the book is toward an introduction to the calculus."

It is a book in which the emphasis is primarily on the understanding of the basic ideas of mathematics rather than on problem solving. Rigorous proofs and definitions are included when possible, since the authors "believe that such treatment enables the student to get a better understanding of the nature of mathematics." These definitions and proofs include the definition of a function as a correspondence rather than by the use of variables, the rigorous analytic definition of a limit and the proofs of several of the limit theorems. The definite integral is first introduced as the least upper bound (or the greatest lower bound) of a certain set of real numbers; it is then proved that this is the limit of the sum.

The book has ample material for two five-semester hour courses but may be used for shorter courses by the elimination of certain sections which are listed in the preface. This is especially useful for students who have had such courses as college algebra and plane trigonometry.

For instructors who accept the viewpoint of the authors regarding the best method of giving students an understanding of the nature of mathematics, this book should prove very useful. It is well written and is rigorously sound. In addition to the instances already given certain areas are treated differently or more briefly since the material is presented for understanding rather than for completeness. Other examples are, the use of the symbols $D_x y$ and $f'(x)$ only for the derivative, the determination of irrational roots of equations of degree greater than three by an approximate method rather than by more systematic methods such as Horner's or Newton's. Also, since the authors believe "that problems play an important role in the assimilation of mathematical ideas" many exercises with illustrative examples are included. However, there is no reference to the historical development of mathematics.

Brief tables of the trigonometric functions and of logarithms are included together with the answers to the odd-numbered problems—JAMES H. ZANT.

The Higher Arithmetic—An Introduction to the Theory of Numbers, H. Davenport. London, W.1.: Hutchinson House, 1952. viii+172 pp., 8s, 6d. net.

In this volume the subject matter of the higher arithmetic deals with the properties of the natural numbers 1, 2, 3, 4, 5, . . . This independent science has been developed within comparatively recent times and doubtless dates from the discoveries of Fermat (1601–65).

Many students are fascinated by the simplicity of numerous theorems whose solutions are suggested by numerical evidence but quite dismayed at the difficulty of proving many of them—thus lending a challenge and enchantment to the study of the higher arithmetic.

While it is stated that the materials of the book are not difficult, the reader will find occasional difficulties unless he has some training in mathematics and facility with it. Readers should find this book interesting and an extension of their knowledge of the topics which are of the type usually presented for an introduction to the theory of numbers.—JOSEPH J. URBANCEK, Chicago Teachers College, Chicago, Illinois.

An Introduction to Mathematical Thought, E. R. Stabler. Cambridge, Mass., Addison-Wesley Publishing Company, Inc., 1953. Cloth, xviii+268 pp., $4.50.

The chief aim of this text, as reported by its author, is " . . . to provide a unified and substantial approach to the logical structure of mathematics, and to develop a corresponding philosophical point of view toward mathematical knowledge." Upon examination of the book, it is evident that the aim is realized. The book is well written. Explanations are clear and easy to follow. A sampling of the exercises indicates that they are interesting and worth-while.

The first part of the book, consisting of five chapters, presents a discussion of the nature of mathematical knowledge and of logical reasoning. An elementary treatment of symbolic logic is presented. This is then applied in a study of the calculus of classes. The second part of the book, consisting of six chapters, treats the postulational methods utilized by modern mathematicians. The application of these methods is illustrated in the development of a finite geometry and in group theory. Prerequisite knowledge for the reading of this book appears to be no more than four years of high school mathematics and the ability to think logically.

The reviewer agrees with the author that the text lends itself both to a course in mathematics intended to give students a concept of the true nature of the subject and some of its cutting edges, and also to a course synthesizing future mathematics teachers' knowledge of the subject. It will also make interesting reading for teachers in service who have intellectual interests.—KENNETH B. HENDERSON, University of Illinois.

An Introduction to Plane Projective Geometry, E. J. Hopkins and J. S. Hails. Oxford, at the Clarendon Press, 1953. vi+276 pp., $5.50.

This text, written for British students, may be useful as a reference, but it is doubtful that it would be as suitable a text for an American college as one by American authors. The treatment is traditional, there being a chapter on homogeneous coordinates and one on the metrical theory of conics; on the whole the treatment is non-metrical.—CECIL B. READ, University of Wichita.

Introduction to the Theory of Statistics, Victor Goedicke. New York, Harper and Brothers, 1953. Cloth, vii+286 pp., $4.50.

According to the Preface, this book is written to give students in non-mathematical fields as much theoretical mastery of statistics as possible through the use of elementary mathematics not involving calculus.

In the first chapter, various statistical problems are proposed for which applicable methods are to be developed in the text. These might well serve as excellent motivation for the student. A great amount of space is devoted to making it appear reasonable to the student that, under certain hypotheses, a statistical variable should have a normal distribution. One unusual feature of the book is the extension of the bar symbol, ordinarily used for the arithmetic mean, to any situation which involves the sum of a set of quantities divided by the number of quantities. The variance, for example, is written as $\sigma^2=\overline{(x-\bar{x})^2}$. The last chapter on the misuses of statistics is exceptionally well done.

Those who feel that mathematical probabilities, in order to be different from 0 or 1, should refer to events that have not yet happened will object to the author's development of probability. Later on in the book, his point of view apparently leads to the interpretation of the probability that a sample will be drawn such that the null hypothesis will not be rejected when it is true, as "the probability that the null hypothesis is correct." When stating confidence intervals, the mean of a specified universe is treated as a statistical variable. Large sample methods are frequently applied to small samples. Sampling distributions of the mean and standard deviation are automatically, without comment, assumed to be normal. One might wish that different notations had been used to distinguish a statistic of the sample from the corresponding parameter of the universe.

In general, it appears that the book is well written, particularly in the motivation for each new topic. It is weakest in the way that the results of tests of hypotheses are stated. This is unfortunate as statistical inference is one of the most important phases of modern statistics.—Fred W. Lott, Iowa State Teachers College, Cedar Falls, Iowa.

Plane Geometry (Revised Edition), Rachel P. Keniston and Jean Tully. Boston, Ginn and Company, 1953. Cloth, v+392 pp., 20 ×25.5 cm., $2.88.

This is a revision of an earlier book by the same authors, but it has been rather thoroughly reworked. The content is so presented that it continually presents views of the future work in mathematics. Space geometry is considered as an extension of two-dimensional geometry and consequently is presented simultaneously with it. This serves to aid the student in obtaining a clearer concept of both geometries. No formal demonstration is done with the space geometry. The text contains many appropriate pictures which are used in the text material. The teacher will be able to expand on these for increased motivation and understanding.

Construction, drawing, and sketching are contrasted. Although the descriptive material emphasizes and discusses the meaning and nature of proof, assumptions, terms, definitions, methods of proof, etc., the teacher will need to be careful that the pupil does not merely follow the illustrative examples as he works.

The first mention of converse is in relation to a definition which has been reversed; the reviewer feels this might lead the pupil to a false concept of converse.

The book is well done and has sufficient material for the most advanced class.—Philip Peak, Indiana University, Bloomington, Ind.

Plane Geometry, A Clear Thinking Approach (Third ed.), Leroy H. Snell and Mildred G. Crawford. New York, McGraw-Hill Book Company, Inc., 1953. xii+436 pp., $3.20.

This text is a revised edition of *Clear Thinking, An Approach through Plane Geometry*. This book suggests experiments for discovering mathematical relationships, some of which are proved deductively later, and emphasizes the application of reasoning learned in geometric situations to everyday problems. A check list is provided for the student to appraise himself of his ability to think clearly. The authors present the idea of arranging facts assumed, or proved to be true, in a notebook where they may be used later, as a carpenter selects tools from a tool box. This is good.

The headings, "Geometry in Life as High School Pupils See It," and, "Everyday Reasoning," appear regularly throughout. "Add a Dimension," carrying the concepts of plane geometry into solid, is frequently presented to excite the interest of pupils who have the ability to study more mathematics. On the other hand, this book could well serve as a text for a combined course of plane and solid geometry. The chapter, "Food for Thought" (Analytic Geometry and Fourth Dimensional Geometry), is also included for the purpose of arousing interest.

This book should be considered by all teachers who are eager that the study of geometry earn an important place in the curriculum of the high school.—Celia H. Canine, High School North, Wichita, Kansas.

Principles of Mathematical Analysis, Walter Rudin. New York, McGraw-Hill Book Company, Inc., 1953. Cloth, ix+227 pp., $5.00.

This well-written book is an up-to-date, concise treatment of the foundations of classical real analysis (real numbers, set theory, limit processes, etc.) and two "modern" topics, the Stone-Weierstrass theorem and Lebesque integration. At the advanced undergraduate or graduate level for which it is intended, it is self-contained. It will serve as a good text for courses at this level and, for those prepared to "dig," it would be a good choice for independent study of the fundamental ideas underlying calculus.—Kenneth O. May, Carleton College, Northfield, Minnesota.

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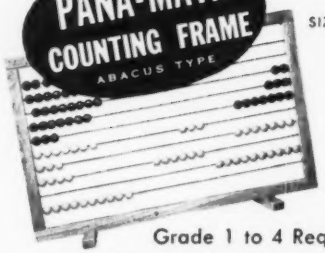

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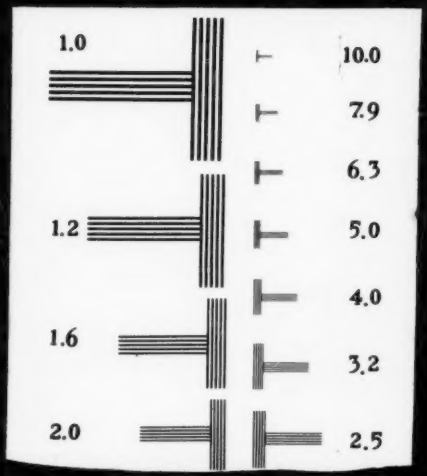

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